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**SECTION 1. Theoretical research in mathematics**

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# ON F-PRIME RINGS AND THEIR F-RINGS OF QUOTIENTS

***Abstract****: All rings are associative in what follows. We will also assume that all rings are unital and all ring homomorphisms preserve the identity. Throughout the paper, B stands for a unitary associative ring and f is an automorphism of B. The main purpose of the work is to describe the left f-ring of quotients $Q_f(B)$ of B. Our methods is not connected with right flat epimorphic hull, but we use a fruitful construction based on a direct limit to build the f-ring of quotients. The principal results to be given are Theorems 1 and 2 below. Theorem 1 establishes that $Q_f(B)$ is embeddable in the complete left ring of quotients $Q_{max}(B)$ and the ring B is a subring of $Q_f(B)$. Theorem 2 asserts that the centres of the left and right f-ring of quotients rings coincides. The authors have intention to continue this study in their subsequent articles. Therefore, at the end of the paper, we formulate a hypothesis, for the proof of which we need the results of this work.*

***Key words****: associative rings; f-rings of quotients; f-prime rings.*
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## Introduction

All rings in this paper are associative with multiplicative identity and all ring homomorphisms are assumed preserve the identity.

*Definition* 1. Let $R$ be a ring and $F$ be an injective ring endomorphism of $R$.

Let $N$ be a subset of $R$. We say that $N$ is an $F$-subset if $F^{-1}(N) = N$. Similarly, we shall say that an ideal $N$ of $R$ is an $F$-ideal if $N$ is an $F$-subset (see [3], [4], [6], [14]). A ring $R$ is said to be $F$-prime if the product of any two nonzero $F$-ideals of $R$ is nonzero. In other words, ring $R$ is $F$-prime if the product of any two $F$-ideals $P$ and $N$ is equal to the zero ideal if and only if either $P = 0$ or $N = 0$.

*Lemma* 1. Let $A$ be a ring and $f$ be an automorphism of $A$. Then the following conditions (1)–(5) are equivalent:

(1) $A$ is $f$-prime;
(2) for any two left $f$-ideals $I$ and $J$ of $A$, the equality $IJ = 0$ implies that either $I = 0$ or $J = 0$;
(3) $a\sum_{i\in\mathbb{Z}} Af^i(b) \neq 0$ for every $a, b \in A$;
(4) $r_A(I) = 0$ for every nonzero $f$-ideals $I$ of $A$;
(5) $\ell_A(I) = 0$ for every nonzero $f$-ideals $I$ of $A$.

## Materials and Methods

Throughout the sequel, $B$ will denote a ring, $f$ will stand for an automorphism of $B$ and we will always assume that the ring $B$ is $f$-prime. We denote by $\Phi(B)$ the set of all nonzero $f$-ideals of $B$.

Despite the fact that rings of rings for along time are classical objects of study, the rings of quotients have been actively studied recently (see, for example, [1], [2]). Now we define the left *f*-ring of quotients of the ring *B*.

We denote by *M* the set of all the pair of the form (*I*, α) where $I \in \Phi(B)$ and $\alpha : I \longrightarrow B$ is a homomorphism of left *B*-modules. Since *B* is *f*-prime, we have $0 \neq IJ \subseteq I \cap J$ for all $I, J \in \Phi(B)$. Определим отношение эквивалентности θ на множестве *M так*: положим (*I*, α) ~ (*J*, β), если существует ненулевой *f*-идеал $K \subset I \cap J$ такой, что $\alpha(x) = \beta(x)$ для всех. Let us define the equivalence relation θ on the set *M* as follows: we put (*I*, α) ~ (*J*, β) if there exists a nonzero *f*-ideal $K \subseteq I \cap J$ such that $\alpha(x) = \beta(x)$ for all $x \in K$. We denote by [*I*, α] the equivalence class containing the pair (*I*, α). Let $Q_f(B) = M/\theta$ (the set of all equivalence classes)

The set $Q_f(B)$ turn into a ring if we define on it the following operations:

$$[I,\alpha] + [J,\beta] = [I \cap J, \alpha + \beta],$$
$$-[I,\alpha] = [I,-\alpha],$$
$$[I,\alpha] \cdot [J,\beta] = [I \cap J, \alpha \circ \beta].$$

*Definition* 2. The ring $Q_f(B)$ described above is called the left (Martindale) *f*-ring of quotients of *B*.

*Remark*. One can formalize the construction of $Q_f(B)$ as the direct limit

$$Q_f(B) = \lim_{\longrightarrow}(\mathrm{Hom}_B({}_BI, {}_BB) : I \in \Phi(B)).$$

If every ideal of *B* is an *f*-ideal, then $Q_f(B)$ coincides with the left ring of quotients in the sense of Martindale.

We define analogously the right *f*-ring of quotients of *B* (in the sense of Martindale) as follows:

$$Q_f^r(B) = \lim_{\longrightarrow}(\mathrm{Hom}_B(I_B, B_B) : I \in \Phi(B)).$$

*Lemma* 2. Let $I \in \Phi(B)$, $J \backslash$ be a left ideal of *B* and $\alpha, \beta : I \longrightarrow B$ be a homomorphism of left *B*-modules. Suppose that $\alpha(x) = \beta(x)$ for all $x \in I \cap J$. Then $\alpha \equiv \beta$.

*Proof.* Take any $b \in J$. If $a \in I$ then $ab \in I \cap J$ and $\alpha(ab) = \beta(ab)$. Hence, $a(\alpha(b) - \beta(b)) = 0$ for all $a \in I$. But $r_A(I) = 0$ by Lemma 1. It follows that $\alpha(b) = \beta(b)$ for all $b \in J$. QED.

The above *f*-ring of quotients $Q_f(B)$ possesses some properties of the usual Martindale ring of quotients. In particular, we have the following theorem.

*Theorem* 1. The *f*-ring of quotients $Q_f(B)$ is embeddable in the maximal left ring of quotients $Q_{max}(B)$ and the ring *B* is a subring of the ring $Q_f(B)$.

*Proof.* If $I \in \Phi(B)$ then $r_B(I) = 0$ by Lemma 1 and, hence, *I* is a dense right ideal of *B*. Therefore, to each element $q = [I, \alpha] \in Q_f(B)$ can be match up with the element $\varphi(q) \in Q_{max}(B))$ such that $x\varphi(q) = \alpha(x)$ for all elements $x \in I$.

One can easily verify that the mapping $q \mapsto \varphi(q)$ is a correctly defined injective ring homomorphism. Furthermore, according to the construction $\varphi(Q_f(B)) = \{q \in Q_f(B)$: there exists an ideal $I \in \Phi(B)$ such that $Iq \subseteq B\}$. Each element $a \in B$ defines the homomorphism of left *B*-modules

$$\alpha_a : x \longmapsto xa \ (x \in B).$$

Since $B \in \Phi(B)$, the equivalence class $[B, \alpha_a]$ belongs to $Q_f(B))$. This gives a natural embedding $B \subseteq Q_f(B)$. QED.

In what follows we identify the rings $Q_f(B)$ and $\varphi(Q_f(B))$.

Let $q \in Q_f(B)$. We denote by *J*(*q*) the sum of all *f*-ideals *I* of *B* such that $Iq \subseteq B$. Observe that *J*(*q*) itself is also an *f*-ideal and $J(q) \cdot q \subseteq B$. Таким образом, $J(q)$ – наибольший идеал среди *f*-идеалов *I* кольца *B*, обладающих свойством $Iq \subseteq B$.

Thus, *J*(*q*) is the largest ideal among *f*-ideals *I* of *B* having the property $Iq \subseteq B$.

If $q \in Q_{max}(B)$ then we set $D(q) = \{a \in Q_{max}(B) : aq \in B\}$. As is well known in the theory of rings of quotients, the automorphism f can be uniquely extended to an automorphism of $Q_{max}(B)$. We will denote this extension by the same symbol $f$. Since $f(D(q)) \cdot f(q) = f(D(q)q) \in f(B) = B$ and $f^{-1}(D(f(q))) \cdot q = f^{-1}(D(f(q)) \cdot f(q)) \in f^{-1}(B) = B$, we get $f(D(q)) = D(f(q))$.

Let us consider any element $q \in Q_f(B)$ and note the following: if $x \in J(q)$ then both $f(x)$ and $f^{-1}(x)$ lie in *J*(*q*), and therefore we have that $xf(q) = f(f^{-1}(x)q) \in B$ and $xf^{-1}(q) = f^{-1}(f(x)q) \in B$. It follows that $J(q)\, f(q) \subseteq B$ and $J(q)\, f^{-1}(q) \subseteq B$. Consequently, $f(Q_f(B)) \subseteq Q_f(B)$ and $f^{-1}(Q_f(B)) \subseteq Q_f(B)$. Thus, $f$ can be regarded as an automorphism of $Q_f(B)$.

*Proposition* 1. The *f*-ring of quotients $Q_f(B)$ introduced above has the following properties.

(1). For any elements $q_1, q_2, \dots, q_n \in Q_f(B)$, there exists an ideal $I \in \Phi(B)$ such that $Iq_i \subseteq B$ for all $i = 1,2, \dots, n$.

(2). If $Iq = 0$ for some ideal $I \in \Phi(B)$ and some element $q \in Q_f(B)$, then $q = 0$.

(3). If $qI = 0$ for some ideal $I \in \Phi(B)$ and some element $q \in Q_f(B)$, then $q = 0$.

(4). If $I \in \Phi(B)$ and $\gamma : I \longrightarrow B$ is a homomorphism of left *B*-modules, then there is an element $q \in Q_f(B)$ such that $\gamma(x) = xq$ for all elements $x \in I$.

(5). $Q_f(B)$ is an *f*-prime ring.

*Proof.* (1). As the required ideal, we can take the ideal

$$I = J(q_1) \cap J(q_2) \cap \ldots \cap J(q_n).$$

It works, because $B$ is an $f$-prime ring and

$$0 \neq J(q_1)J(q_2) \cdots J(q_n) \subseteq J(q_1) \cap J(q_2) \cap \ldots \cap J(q_n).$$

(2). This follows from Lemma 2 and the definition of the $f$-ring of quotients.

(3). Observe that $P = \sum_{i\in\mathbb{Z}} J(q)f^i(q)$ is the left $f$-ideal of $B$ and $PI = \sum_{i\in\mathbb{Z}} J(q)f^i(qI) = 0$. Since is an $f$-prime ring, then $P = 0$. Therefore, $J(q)q = 0$ and $q = 0$.

(4). This assertion follows immediately from Definition 2.

(5). Let $a$ and $b$ be two non-zero elements of $Q_f(B)$. Then it follows from (2) that $\sum_{i\in\mathbb{Z}} J(a)f^i(a)$ and $\sum_{i\in\mathbb{Z}} J(b)f^i(b)$ are non-zero left $f$-ideals. Since $B$ is an $f$-prime ring, we get that $\sum_{i\in\mathbb{Z}} J(a)f^i(a) \cdot \sum_{i\in\mathbb{Z}} J(b)f^i(b) \neq 0$. Hence, it follows that $a \cdot \sum_{i\in\mathbb{Z}} J(b)f^i(b) \neq 0$ and, by Lemma 1, $Q_f(B)$ is an $f$-prime ring. QED.

We denote by $C(B)$ and $C^r(B)$ the center of the left maximal and right maximal rings of quotients of B, correspondingly. One more piece of notation: $C_f(B) = \{c \in C(B) : f(b) = b\}$ and $C_f^r(B) = \{c \in C^r(B) : f(b) = b\}$.

*Theorem* 2.

(1). If $q \in C_f(B)$, then the set $D(q) = \{b \in B \mid bq \in B\}$ is an f-ideal.

(2). $C_f(B)$ is the center of $Q_f(B)$.

(3). If $q = [I, \alpha] \in Q_f(B)$, then $q \in C_f(B)$ if and only if α is a homomorphism of bimodules over the ring $B$.

(4). The rings $C_f(B)$ and $C_f^r(B)$ are isomorphic.

*Proof.* (1). Let $a \in B$. Then $qa \in B$ if and only if $f(qa) = f(a)\,q \in B$. Therefore, $D(q) \in \Phi(B)$.

Assertion (2) follows from assertion (1) since $C_f(B)$ is the centralizer of the set $B$ in $Q_{max}(B)$.

(3). If α is a homomorphism of bimodules over the ring $B$, then, as is easily seen, $q \in C(B)$ and therefore $q \in C_f(B)$. The converse follows from assertion (1).

(4). Using assertion (3), one can prove that both rings $C_f(B)$ and $C_f^r(B)$ are isomorphic to the ring

$$\{c \in \varinjlim(\mathrm{Hom}_B({}_BI_B, {}_BB_B) : I \in \Phi(B)) \mid f(c) = c\}.$$

QED.

*Theorem* 3. Let $B$ be an $f$-prime ring and $0 \neq q \in Q_f(B)$. If $q\,f(a) = aq$ for all elements $a \in B$, then $q$ is an invertible element of the ring $Q_f(B)$ and $f$ is an inner automorphism of the ring $Q_f(B)$ defined by the element $q$:

$$f(p) = q^{-1}pq \quad (\forall p \in Q_f(B)).$$

*Proof.* For brevity, we denote by $Q$ the ring of quotients $Q_f(B)$. Let $p$ be an arbitrary element of the ring Q and let k be an integer. Then for each elements $a \in B$, we have a chain of equalities:

$$f^k(q)f(a) = f^k\left(qf^{1-k}(a)\right) = f^k(f^{-k}(a)q) = = af^k(q). \quad (1)$$

We recall that $J(p)$ denotes the largest among the $f$-ideals $I$ of $B$ with the property $Ip \subseteq B$. It follows from (1) that for all elements $a \in J(p)$ the following equalities hold:

$$af^k(q)f(p) = f^k(q)f(a)f(p) = f^k(q)f(ap) = apf^k(q).$$

Hence $J(p)\left(f^k(q)f(p) - pf^k(q)\right) = 0$.

Therefore, $f^k(q)f(p) - pf^k(q) = 0$ by Proposition 1. Consequently,

$$f^k(q)f(p) = pf^k(q) \quad (\forall p \in Q\ \forall k \in \mathbb{Z}) \quad (2)$$

and in particular,

$$f^k(q)Q = Qf^k(q) \quad (3)$$

The ring $Q$ is $f$-prime by Proposition 1. Therefore, it follow from Lemma 1 that $a\sum_{i\in\mathbb{Z}} Qf^i(b) \neq 0$ for any two non-zero elements $a$, $b \in Q$. Hence, $q\sum_{i\in\mathbb{Z}} f^i(q)Q = q\sum_{i\in\mathbb{Z}} Qf^i(q) \neq 0$ by (3), and therefore $qf^k(q) \neq 0$ for some integer $k$. We set $u = qf^k(q)$ and note that, by virtue of (2), we obtain that

$$pu = pqf^k(q) = qf(p)f^k(q) = qf^k(q)f^2(p) = uf^2(p) \quad (4)$$

for all $p \in Q$. Moreover, equality (1) leads to the invariance of the element $u$:

$$u = qf^k(q) = f^k(q)f(q) = f(q)f(f^k(q)) = f(u).$$

So, we have proved that

1) $uQ = Qu$ (this follows from equality (4));

2) $f(u) = u$.

To complete the proof, we need the following lemma.

*Lemma* 3 ([9, Lemma 1.2]). If an element $u \in Q$ satisfies conditions 1) and 2), then $u$ is an invertible element in the ring $Q$.

To finish the proof, it remains to note that $q$ also turns out to be an invertible element of the ring $Q$. Indeed, $qf^k(q)u^{-1} = uu^{-1} = 1$ and $u^{-1}f^{k-1}(q)q = u^{-1}qf^k(q) = u^{-1}u = 1$. It follows from (2) that $q\,f(p) = pq$ for all elements $p \in Q$. Since $q$ is invertible, $f(p) = q^{-1}pq$ $(\forall p \in Q)$. QED.

Now we formulate an open problem:

Let $R$ be a ring an $f$-prime ring, and $A$ be the Cohn-Jordan extension of $R$ by means of $f$ (see [10], [11], [12], [14]). The one can prove that $A$ is also $f$-prime. It would be interesting to clarify the relationship between the $f$-rings of quotients of rings $R$ and $A$.

**Conclusion**

At the end of the present work, let us make the following hypothesis, which will be verified in the subsequent author's works:

Let $R$ be a ring and F be an injective endomorphism of $R$. Suppose that the skew polynomial ring $R[x, F]$ is semiprime. Let $(A, f)$ be the Cohn-Jordan extension of the pair $(R, f)$. This is extension described in [3], [4], and [5]. We denote by $Q$ the maximal left ring of quotients of $A$ and denote by $\mathfrak{D}_*$ the orthogonal completion of the center of the ring of oblique Laurent polynomials $Q\langle x, f\rangle$ in the maximal left ring of quotients of $Q\langle x, f\rangle$. Then the extended centroid of the ring $R\,[x, F]$ is isomorphic to the complete left classical ring of quotients $\mathfrak{D}_*^{-1}\mathfrak{D}_*$.

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