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Ruslan Malikov
Samarkand State University
Researcher

Gulnoza Abdirashidova
Samarkand State Medical Institute
Researcher

Ablakul Abdirashidov
Samarkand State University
Corresponding member of International Academy,
Doctor of Physical and Mathematical Sciences,
Professor to department of theoretical and applied mechanics
abdira@mail.ru

APPROXIMATE SOLUTION SOME DELAY DIFFERENTIAL EQUATIONS USING COMBINATION OF THE LAPLACE TRANSFORM AND THE VARIATIONAL ITERATION METHOD

Abstract: This paper is concerned with obtaining the exact and approximate solutions of linear and nonlinear delay differential equations via a combination of the Laplace transform and variational iteration method. In this approach, a correction functional is constructed by a general Lagrange multiplier, which is determined by using the Laplace transform with the variational theory. Examples are given to elucidate the exact and approximate solution process, the simplicity, efficiency and reliability of this approach.

Key words: delay differential equations, variational iteration method, Laplace transform, inverse Laplace transform, Lagrange multiplier, approximate solution.

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Introduction

Nonlinear phenomena are of fundamental importance in various fields of science and technology. Nonlinear models of real-world problems are still difficult to solve either numerically or theoretically. Recently, much attention has been paid to the search for better and more efficient approximate or exact, analytical or numerical methods for solving for nonlinear models [1, 2, 5, 7, 8]. There are many standard semi-analytical methods for solving linear and nonlinear delay differential equations (DDE), for example, the Adomian decomposition method, the

variational iterations method and their various modifications [3-6, 9].

DDE arises when the rate of change of an unsteady process during its mathematical modeling is determined not only by its current state, but also in a certain past state known as its history. The introduction of delays in models enriches the dynamics of such models and allows us to accurately describe the phenomena of real life. DDE often occur in many mechanical, physical, and biomedical phenomena. In particular, they are fundamental when ordinary differential equations (ODE) do not work. Unlike ODEs, in which initial conditions are specified

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at the initial point, DDE requires a system history for the delayed interval, and they are specified as initial conditions. For this reason, delay systems are complex. Because of this complexity, DDE is difficult to analyze analytically and therefore requires an approximate or numerical approach [8, 10, 11].

The Adomian decomposition method and the variational iterations method is one of the well-known methods for solving various linear and nonlinear evolution equations. Many studies have proven that these methods are reliable and effective for a wide range of scientific applications, linear and nonlinear equations with bounded and unbounded domains [1-7]. These methods have no special requirements, such as linearization, small parameters, and so on for nonlinear operators. Below, the Cauchy problem with the different order linear and nonlinear ordinary differential equations are solved analytically using the Laplace variational iterations method.

VIM has been successfully applied to many initial and regional tasks. Steps of application of VIM to differential equations: obtaining the correction functional; identifying the Lagrange multiplier; determining a good initial approximation. This algorithm was proposed in [8,10].

Various authors have identified this Lagrange multiplier via different approaches in order to accelerate the convergence rate of solutions [7-11]. In [8, 10], it was proposed to identify the Lagrange multiplier using the Laplace transform and the variational theory. The basic motivation of this paper is the extension of this approach for solving linear and nonlinear delay differential equations which are otherwise difficult to analyze because of their complex nature and infinite dimensionality.

Analysis of the methods.

1) Variational iterations method.

The basic steps involved are given as follows:

Considering the differential equation below in an operator form as

$$Ry(t) + Ny(t) = f(t) \quad (1)$$

where R is linear operator, N is a nonlinear operator and $f(t)$ is known analytical function.

$$y_{n+1}(t) = y_n(t) - L^{-1} \left\{ \frac{1}{s^m} \left[s^m Y_n(s) - s^{m-1} y(0) - \dots - y^{(m-1)}(0) - L(Ry_n(t) + Ny_n(t) - f(t)) \right] \right\} = L^{-1} \left[\frac{y(0)}{s} + \dots + \frac{y^{(m-1)}(0)}{s^m} \right] + L^{-1} \left\{ \frac{1}{s^m} L[Ry_n(t) + Ny_n(t) - f(t)] \right\},$$

with initial approximation

$$y_0(t) = L^{-1} \left[\frac{y(0)}{s} + \dots + \frac{y^{(m-1)}(0)}{s^m} \right] = y(0) + y'(0)t + \dots + \frac{y^{(m-1)}(0)}{(m-1)!} t^{m-1}.$$

According to the variational iterations method, we can construct a correction functional as follows:

$$y_{n+1}(t) = y_n(t) + \int_0^t \lambda(s) [Ry_n(s) + N\tilde{y}_n(s) - f(s)] ds \quad (2)$$

where λ is a general Lagrange multiplier, which can be identified optimally via the variational theory (He, 2007), the subscript n denotes the n th approximation and \tilde{y}_n is considered as a restricted variation, i.e., $\delta\tilde{y}_n = 0$. It is obvious now that the main steps of the variational iterations method require first the determination of the Lagrangian multiplier λ that will be identified optimally. Having determined the Lagrangian multiplier, the successive approximations $y_{n+1}(t)$, $n \geq 0$, of the solution u will be readily obtained upon using any selective function $y_0(t)$.

Consequently, the solution $y(t) = \lim_{n \rightarrow \infty} y_n(t)$.

Lagrange multiplier can be easily identified as:

$$\lambda(s) = (-1)^m \frac{1}{(m-1)!} (s-t)^{m-1}.$$

2) Laplace Variational Iteration Method.

This algorithm (Laplace variational iteration method) was proposed in [7, 8, 10]. The basic steps involved are given as follows:

Take the Laplace transform of (1), then the correction functional is

$$Y_{n+1}(s) = Y_n(s) + \lambda(s) [s^m Y_n(s) - s^{m-1} y(0) - \dots - y^{(m-1)}(0) + L(R\tilde{y}_n(t) + N\tilde{y}_n(t) - f(t))]. \quad (3)$$

Regarding the terms $L[R\tilde{y}_n(t) + N\tilde{y}_n(t)]$ as restricted variations, we make (3) stationary with respect to $Y_n(t)$

$$\delta Y_{n+1}(s) = \delta Y_n(s) + \lambda(s) [s^m \delta Y_n(s)]$$

From this we define the Lagrange multiplier as

$$\lambda(s) = -1/s^m.$$

The successive approximations are obtained by taking the inverse Laplace transform to obtain

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Numerical Applications.

Below, the above procedure is applied to the solution of certain linear and nonlinear delay different equations. The solution to these problems is carried out using Maple 17.

$$y'(t) = -2 \sin\left(\frac{t}{2}\right) y\left(\frac{t}{2}\right), \quad t \in [0, 1], \quad y(0) = 1,$$

The exact solution is given by $y(t) = \cos t$.

Taking the inverse Laplace transform, we obtain the iteration formula

Example 3.1. We consider the first order linear DDE

$$Y_{n+1}(s) = Y_n(s) + \lambda(s) \left\{ sY_n(s) - y(0) - L \left[-2 \sin\left(\frac{t}{2}\right) y_n\left(\frac{t}{2}\right) \right] \right\},$$

where $\lambda(s)$ Lagrange multiplier: $\lambda(s) = -1/s$. Taking the inverse Laplace transform, we obtain

with initial approximation $y_0(t) = 1$. From this iterative formula we obtain

$$y_{n+1}(t) = 1 + L^{-1} \left\{ \frac{1}{s} L \left[-2 \sin\left(\frac{t}{2}\right) y_n\left(\frac{t}{2}\right) \right] \right\},$$

$$y_1(t) = -3 + 4 \cos\left(\frac{t}{2}\right);$$

$$y_2(t) = \frac{11}{3} + \frac{64}{3} \cos^3\left(\frac{t}{4}\right) - 24 \cos^2\left(\frac{t}{4}\right);$$

$$y_3(t) = \frac{419}{35} - \frac{352}{3} \cos^2\left(\frac{t}{8}\right) + \frac{1504}{3} \cos^4\left(\frac{t}{8}\right) - \frac{4096}{15} \cos^5\left(\frac{t}{8}\right) - 512 \cos^6\left(\frac{t}{8}\right) + \frac{8192}{21} \cos^7\left(\frac{t}{8}\right);$$

...

After the fifth iteration, the maximum absolute error is less than 10^{-10} . This sequence converges to the exact solution as $n \rightarrow \infty$.

Example 3.2. We consider the first order nonlinear DDE

$$y'(t) = 1 - 2y^2\left(\frac{t}{2}\right) + y(t) - \sin(t), \quad t \in [0, 1], \quad y(0) = 1,$$

The exact solution is given by $y(t) = \cos t$.

Taking the inverse Laplace transform, we obtain the iteration formula

$$Y_{n+1}(s) = Y_n(s) + \lambda(s) \left\{ sY_n(s) - y(0) - L \left[1 - 2y_n^2\left(\frac{t}{2}\right) + y_n(t) - \sin t \right] \right\},$$

where $\lambda(s)$ Lagrange multiplier: $\lambda(s) = -1/s$. Taking the inverse Laplace transform, we obtain

$$y_{n+1}(t) = 1 + L^{-1} \left\{ \frac{1}{s} L \left[1 - 2y_n^2\left(\frac{t}{2}\right) + y_n(t) - \sin t \right] \right\},$$

with initial approximation $y_0(t) = 1$. From this iterative formula we obtain

$$y_1(t) = \cos t; \quad y_2(t) = \cos t; \quad y_3(t) = \cos t; \dots$$

Thus, an exact solution is obtained.

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Example 3.3. Consider the second order linear DDE

$$y''(t) = -y(t) + y\left(t - \frac{1}{2}\right) + t + 1.75, \quad t \in [0, 1], \quad y(0) = 0, \quad y'(0) = 0$$

The exact solution is given by $y(t) = t^2$.

Taking the inverse Laplace transform, we obtain the iteration formula

$$Y_{n+1}(s) = Y_n(s) + \lambda(s) \left\{ s^2 Y_n(s) - sy(0) - y'(0) - L \left[-y_n(t) + y_n\left(t - \frac{1}{2}\right) + t + 1.75 \right] \right\},$$

where $\lambda(s)$ Lagrange multiplier: $\lambda(s) = -1/s^2$.
Taking the inverse Laplace transform, we obtain

$$y_{n+1}(t) = L^{-1} \left\{ \frac{1}{s^2} L \left[-y_n(t) + y_n\left(t - \frac{1}{2}\right) + t + 1.75 \right] \right\},$$

with initial approximation $y_0(t) = 0$. From this iterative formula we obtain

$$\begin{aligned} y_1(t) &\approx 0.875t^2 + 0.16667t^3; \\ y_2(t) &\approx 0.97396t^2 + 0.04167t^3 - 0.02083t^4; \\ y_3(t) &\approx 0.99349t^2 + 0.01128t^3 - 0.00781t^4 + 0.00208t^5; \\ y_4(t) &\approx 0.9982t^2 + 0.00326t^3 - 0.0026t^4 + 0.00104t^5 - 0.00017t^6; \dots \end{aligned}$$

After the ten iteration, the maximum absolute error is less than 10^{-6} . This sequence converges to the exact solution as $n \rightarrow \infty$.

$$y''(t) = 1 - 2y^2\left(\frac{t}{2}\right), \quad t \in [0, 1], \quad y(0) = 1, \quad y'(0) = 0$$

The exact solution is given by $y(t) = \cos t$.
Taking the inverse Laplace transform, we obtain the iteration formula

Example 3.4. Consider the second order nonlinear DDE

$$Y_{n+1}(s) = Y_n(s) + \lambda(s) \left\{ s^2 Y_n(s) - sy(0) - y'(0) - L \left[1 - 2y_n^2\left(\frac{t}{2}\right) \right] \right\},$$

where $\lambda(s)$ Lagrange multiplier: $\lambda(s) = -1/s^2$.
Taking the inverse Laplace transform, we obtain

$$y_{n+1}(t) = 1 + L^{-1} \left\{ \frac{1}{s^2} L \left[1 - 2y_n^2\left(\frac{t}{2}\right) \right] \right\},$$

with initial approximation $y_0(t) = 1$. From this iterative formula we obtain

$$\begin{aligned} y_1(t) &= 1 - \frac{1}{2}t^2; \\ y_2(t) &= 1 - \frac{1}{2}t^2 + \frac{1}{24}t^4 - \frac{1}{960}t^6; \\ y_3(t) &= 1 - \frac{1}{2}t^2 + \frac{1}{24}t^4 - \frac{1}{720}t^6 + \frac{1}{40980}t^8 - \frac{1}{4147200}t^{10} + \frac{1}{778567680}t^{12} - \frac{1}{343513497600}t^{14}; \\ &\dots \end{aligned}$$

After the fifth iteration, the maximum absolute error is less than 10^{-15} . This sequence converges to the exact solution as $n \rightarrow \infty$.

Example 3.5. Consider the third order linear DDE

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$$y'''(t) = y(t) + y(t-1) - e^{t-1}, \quad t \in [0,1], \quad y(0) = 1, \quad y'(0) = 1, \quad y''(0) = 1.$$

The exact solution is given by $y(t) = e^t$.

Taking the inverse Laplace transform, we obtain the iteration formula

$$Y_{n+1}(s) = Y_n(s) + \lambda(s) \left\{ s^3 Y_n(s) - s^2 y(0) - sy'(0) - y''(0) - L \left[y_n(t) + y_n(t-1) - e^{t-1} \right] \right\},$$

where $\lambda(s)$ Lagrange multiplier: $\lambda(s) = -1/s^3$.

Taking the inverse Laplace transform, we obtain

$$y_{n+1}(t) = 1 + t + \frac{t^2}{2} + L^{-1} \left\{ \frac{1}{s^3} L \left[y_n(t) + y_n(t-1) - e^{t-1} \right] \right\},$$

with initial approximation $y_0(t) = 1 + t + \frac{t^2}{2}$.

After the ten iteration, the maximum absolute error is less than 10^{-5} . This sequence converges to the exact solution as $n \rightarrow \infty$.

Example 3.6. Consider the third order nonlinear DDE

$$y'''(t) = y(t) - y^2(t-1) + f, \quad t \in [0,1], \quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 2,$$

where $f = t^6 - 4t^5 + 6t^4 - 5t^3 + 6$. The exact solution is given by $y(t) = t^3 + t^2$.

Taking the inverse Laplace transform, we obtain the iteration formula

$$Y_{n+1}(s) = Y_n(s) + \lambda(s) \left\{ s^3 Y_n(s) - s^2 y(0) - sy'(0) - y''(0) - L \left[y_n(t) - y_n^2(t-1) + f \right] \right\},$$

where $\lambda(s)$ Lagrange multiplier: $\lambda(s) = -1/s^3$.

Taking the inverse Laplace transform, we obtain

$$y_{n+1}(t) = t^2 + L^{-1} \left\{ \frac{1}{s^3} L \left[y_n(t) - y_n^2(t-1) + f \right] \right\},$$

with initial approximation $y_0(t) = t^2$.

After the ten iteration, the maximum absolute error is less than 10^{-5} . This sequence converges to the exact solution as $n \rightarrow \infty$.

Conclusion.

Thus, the approach used, with the Laplace transform and VIM, allows obtaining exact /

approximate solutions of linear and nonlinear delay differential equations. The Lagrange multipliers used in this approach are easily identified and allow one to obtain new variational iterative formulas. This algorithm is used without using linearization, discretization, or unrealistic assumptions. The method provides more realistic consistent solutions that converge very quickly in physical, medical and biological problems. This method is able to reduce the amount of computational work compared with classical methods, while maintaining high accuracy of the numerical result. The approach with the solution of some DDEs shows that this method is very simple, accurate and efficient [2, 8-14].

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