Impact Factor:

ISRA (India)	= **6.317**	**SIS** (USA)	= **0.912**	**ICV** (Poland)	= **6.630**
ISI (Dubai, UAE)	= **1.582**	**РИНЦ** (Russia)	= **0.126**	**PIF** (India)	= **1.940**
GIF (Australia)	= **0.564**	**ESJI** (KZ)	= **9.035**	**IBI** (India)	= **4.260**
JIF	= **1.500**	**SJIF** (Morocco)	= **7.184**	**OAJI** (USA)	= **0.350**

SOI: 1.1/TAS DOI: 10.15863/TAS

International Scientific Journal

Theoretical & Applied Science

p-ISSN: 2308-4944 (print) **e-ISSN:** 2409-0085 (online)

Year: 2021 **Issue:** 08 **Volume:** 100

Published: 11.08.2021 **http://T-Science.org**

QR – Issue **QR – Article**

Zakaria Yahia Hussein Halema
Shaqra University
Department of Mathematics
Kingdom of Saudi Arabia
halimazakariaa@gmail.com

FINITE DIMENSIONAL APPROXIMATION AND WIENER MEASURE

***Abstract**: We introduce the notion of the Finite Dimensional Approximation and Wiener Measure, Differential Calculus on the Riemannian Path Space and The maps $\rho\mathcal{P}$ and $\tilde{\rho}\mathcal{P}$ The matrix $G_{\mathcal{P}}^{\mathcal{F}_{\mathcal{P}}}$. We prove the finite dimensional version of the intertwinning formula for the derivative Theorem (6.1.8) [1] and the finite dimensional integration by parts formula Theorem (3.2).*

***Key words**: Riemannian manifold, Itô filtration, path space, Ornstein–Uhlenbeck operator, Gaussian measure, Wiener measures, Brownian motion.*

***Language**: English*

Citation: Halema, Z. Y. H. (2021). Finite Dimensional Approximation And Wiener Measure. *ISJ Theoretical & Applied Science, 08 (100),* 101-107.

Soi: http://s-o-i.org/1.1/TAS-08-100-20 ***Doi***: crossref https://dx.doi.org/10.15863/TAS.2021.08.100.20

***Scopus ASCC**: 2600.*

Introduction

In [2] the foundations of a Riemannian geometry on $W(M)$ have been defined (cf. also [3]). This leads to an extension of the initial notion of tangent space.This work is an attempt to approximate systematically the geometrical objects on the path space by finite dimensional ones. This procedure justifies a posteriori and in some sense the choice of the Markovian connection. In particular, it allows to construct a process on the frame bundle of the path space which corresponds to the lift of the Ornstein–Ulhenbeck–Driver–Röckner process. The lifted process plays a crucial role in the development of the stochastic calculus of variations on the path space [4]. For other finite dimensional approaches to analysis and geometry on path spaces we refer to [5,6,7,8].

We give the basic definitions of differential geometry on the path space. We construct the finite dimensional geometrical objects based on finite partitions of the time interval and in particular we construct a discretized version of Markovian connection. We prove the finite dimensional version of the intertwinning formula for the derivative (Theorem (6.1.8)) and the finite dimensional integration by parts formula (Theorem (6.1.11)). As the mesh of the partitions goes to zero, we derive in an independent way, correspondingly, statement 2.6 of [2] and Bismut's formula. We devoted to finite dimensional approximation of the Ornstein–Uhlenbeck operator, associated process and corresponding semigroup.

Let $(M,\langle\cdot,\cdot\rangle_m)$ be a compact Riemannian manifold of dimension d, where $\langle\cdot,\cdot\rangle_m$ is the Riemannian metric. On this Riemannian manifold we consider the Levi–Civita connection associated with $\langle\cdot,\cdot\rangle_m$. Let $O(M)$ denote the bundle of orthogonal frames over M, namely

$$O(M) := \{(m,r): r \text{ is a Euclidean isometry from } \mathbb{R}^d \text{ into } T_m(M), m \in M\}.$$

Then $O(M)$ is a parallelized manifold.

Definition (1.1): (Tangent Space**):** Let $\sigma \in H^{(n)}(M), b = I_n^{-1}(\sigma), r$ the horizontal lift of σ. For $z \in H$, then $Z(s) := t_{s\leftarrow 0}^{\sigma}\big(z(s)\big) \in T_\sigma H^{(n)}(M)$ if and only if

$$\ddot{z}(s) = \Omega_{r(s)}\left(\dot{b}(s), z(s)\right)\dot{b}(s) \quad on \quad I\setminus\mathrm{P}.$$

This tangent space is inherited from the tangent space of the Gaussian vector space $H^{(n)}$ through the

Itô mapping I_n (cf. [6, Proposition 4.4]).

Let $M^{(n)} := M^n$ and $\pi_n^W : W(M) \longmapsto M^{(n)}$ $\left(resp.\ \pi_n^X : X \longmapsto H^{(n)}\right)$ denote the projection

$$\pi_n^W(p) := \left(p(s_1), \dots, p(s_n)\right);$$
$$\left(resp.\ \pi_n^X(b) := \left(b(s_1), \dots, b(s_n)\right)\right).$$

On this space there is a natural tangent space $T\left(M^{(n)}\right) = T(M)^n$ which is different from the previous one on $H^{(n)}(M)$. We shall establish a relation between these two tangent spaces.

We can endow $H^{(n)}(M)$, with a Gaussian measure ν_n such that $\nu_n \circ I_n = \mu_n$, where $\mu_n = \mu \circ (\pi_n^X)^{-1}$ is the finite dimensional Gaussian measure on $H^{(n)}$.

For $\varepsilon \in [0,1]$, let

$$M_\varepsilon^{(n)} := \left\{v \in M^{(n)} : d(v_i, v_{i+1}) < \zeta_\varepsilon,\quad for\quad i = 0,1, \dots, n-1\right\},$$
$$H_\varepsilon^{(n)}(M) := \left\{\sigma \in H^{(n)}(M) : \int_{s_i}^{s_{i+1}} |\dot\sigma(s)| ds < \zeta_\varepsilon,\quad for\quad i = 0,1, \dots, n-1\right\},$$
$$H_\varepsilon^{(n)} := \left\{z \in H^{(n)} : \|\Delta_i z\| < \zeta_\varepsilon,\quad for\quad i = 0,1, \dots, n-1\right\},$$

where

$$\zeta_\varepsilon := \varepsilon(\rho \wedge 4/K_\Omega) \tag{1.1}$$

and ρ is the injectivity radius of M, $K_\Omega = sup_{r \in O(M)} \|\Omega_r\| < \infty$.

$M_\varepsilon^{(n)}$ is an open subset of $M^{(n)}$ and therefore is a differentiable manifold. We associate to $v \in M_\varepsilon^{(n)}$ the piecewise geodesic curve σ_v defined by linking the points v_i, v_{i+1} by the minimizing geodesic. For $v \in M_\varepsilon^{(n)}$, we consider the map

$$\left[\Theta_v^{(n)}\right]^{-1} : H^{(n)} \longmapsto T_v\left(M_\varepsilon^{(n)}\right)$$

given by $Z(s_i) := t_{s_i \leftarrow 0}^{\sigma_v}\left(z(s_i)\right) \in T_{v_i}(M)$, $i = 1, \dots, n$, where $z \in H^{(n)}$. Then $\Theta^{(n)}$ defines a parallelism on $M_\varepsilon^{(n)}$.

Definition (1.2): For any smooth vector fields $Y, Z \in T\left(M_\varepsilon^{(n)}\right)$, put

$$\frac{d}{ds}\left(\nabla_Y^{(n)} Z\right)^\alpha (v, s^-) := \mathrm{D}_Y^{(n)} \dot z^\alpha(s^-)$$
$$+ \int_0^{s^-} \Omega_{\gamma\lambda\beta}^\alpha\left(\sigma_v(\tau)\right) y^\gamma(\tau) d[I_n^{-1}(\sigma_v)]^\lambda(\tau) \cdot \dot z^\beta(s^-).$$

Here for $f \in C^\infty\left(M_\varepsilon^{(n)}\right)$

$$\left(\mathrm{D}_{s,\alpha}^{(n)} f\right)(v) := \sum_{k=1}^n 1_{s<s_k} \langle t_{0 \leftarrow s_k}^{\sigma_v} \partial_k f, \varepsilon_\alpha \rangle_{m_0}$$

and

$$\mathrm{D}_Y^{(n)} f := \int_0^1 \mathrm{D}_{s,\alpha}^{(n)} f \cdot \dot y^\alpha(s)\, ds = \sum_{k=1}^n \langle \partial_k f, Y(s_k) \rangle_{v_k} = Yf.$$

Heuristic path integrals such as those in Eq. (6.48)[12] have proven themselves useful and arise often in physics literature. Particularly, one can interpret this path integral as the path integral quantization of the Hamiltonian on M. Much of the current interest concerning path integrals in physics began with Feynman in [13] and has since grown deeply. The role of path integrals in quantum mechanics is surveyed by Gross in [14] and detailed more by Feynman and Hibbs in [15] as well as Glimm and Jaffe in [16].

For the partition $\mathcal{P} = \{0 = s_0 < s_1 < \cdots < s_n = 1\}$, define the finite-dimensional subspace $H_\mathcal{P}(M)$ of $W(M)$ by

$$H_\mathcal{P}(M) = \{\sigma \in W(M) : \sigma \text{ is piecewise geodesic with respect to } \mathcal{P}\}. \tag{1.2}$$

We make $H_\mathcal{P}(M)$ into a Riemannian manifold by endowing it with the L^2 metric $G_\mathcal{P}$, defined by

$$G_\mathcal{P}(X, Y) = \int_0^1 g\left(X(s), Y(s)\right) ds, \tag{1.3}$$

where we are making the natural identification of the tangent space $T_\sigma H_\mathcal{P}(M)$ with the piecewise Jacobi fields along σ in M. From here we define the approximate Wiener measure $\nu_{G_\mathcal{P}}$ on $H_\mathcal{P}(M)$ by

$$d\nu_{G_\mathcal{P}} = \frac{1}{Z_{G_\mathcal{P}}} e^{-\frac{1}{2}\int_0^1 \|\sigma'(s)\|^2 ds} dVol_{G_\mathcal{P}}, \tag{1.4}$$

where $Vol_{G_\mathcal{P}}$ is the Riemannian volume form given by $G_\mathcal{P}$ and $Z_{G_\mathcal{P}}$ is a normalization constant which forces $\nu_{G_\mathcal{P}}$ to be a probability measure in the case that $M = \mathbb{R}^d$. With the matrix $\mathcal{L}_\mathcal{P}$ introduced below in Eq. (6.97)[12],

$$Z_{G_\mathcal{P}} = \sqrt{det\ \mathcal{L}_\mathcal{P}} \prod_{i=1}^n (2\pi\Delta_i s)^{d/2}. \tag{1.5}$$

It is well known that the Wiener measure on $W(\mathbb{R}^d)$ is the law of an $\mathbb{R}^d$-valued Brownian motion, and conversely, the evaluation maps $b_s(\omega) = \omega(s)$ on $W(\mathbb{R}^d)$ are an $\mathbb{R}^d$-valued Brownian motion under the Wiener measure.

In what follows we use the symbols μ and ν to denote the Wiener measures on $W(\mathbb{R}^d)$ and $W(M)$ respectively. Although we will consider several probability spaces, the symbol $\mathbb{E}$ will be used solely for expectation on the probability space $(W(\mathbb{R}^d), \mu)$.

The piecewise approximation of Brownian motion with respect to the partition $\mathcal{P}$ are the maps $b_s^\mathcal{P} : W(\mathbb{R}^d) \to H_\mathcal{P}(\mathbb{R}^d)$ with $s \in [0, 1]$ given by

$$b_s^{\mathcal{P}} := \sum_{i=1}^{n} 1_{J_i}(s)\left[\frac{\Delta_i b}{\Delta_i s}(s - s_{i-1}) + b_{s_{i-1}}\right]. \quad (1.6)$$

Here and forevermore $\Delta_i b = b_{s_i} - b_{s_{i-1}}$, and $J_i = (s_{i-1}, s_i]$ when $i > 1$ and $J_1 = [0, s_1)$. It is important to note that $b_s|_{H_{\mathcal{P}}(\mathbb{R}^d)} = b_s^{\mathcal{P}}|_{H_{\mathcal{P}}(\mathbb{R}^d)}$.

This is a convenient place to introduce the Cameron–Martin subspace $H(M)$ of the Wiener space, which is the collection of absolutely continuous paths with finite energy,

$$H(M) = \left\{\sigma \in W(M): \sigma \text{ is absolutely continuous}, \int_0^1 \|\sigma'(s)\|^2 ds < \infty\right\}. \quad (1.7)$$

The Cameron–Martin space is a Hilbert space and $(i, H(\mathbb{R}^d), W(\mathbb{R}^d))$ is the prototype for an abstract Wiener space, where $i : H(\mathbb{R}^d) \to W(\mathbb{R}^d)$ is the canonical injection.

The tangent space $T_\sigma H_{\mathcal{P}}(M)$ is identified with the continuous piecewise Jacobi fields along σ.

We introduce the measure $v_{S_{\mathcal{P}}}$ in Eq. (6.60) and $\mu_{S_{\mathcal{P}}}$, where $\mu_{S_{\mathcal{P}}}$ is simply realization of $v_{S_{\mathcal{P}}}$ in the flat case $M = \mathbb{R}^d$.

II. Preliminaries

We consider the orthogonal basis of $H^{(n)}$ defined in Theorem (6.1.24)[1], and the corresponding parallelized vector fields $H_{i,\alpha}(v,s) = t_{s\leftarrow 0}^{\sigma_v} h_{i,\alpha}$.

We denote by $B_{i,\alpha}$ the horizontal lift of $H_{i,\alpha}$ through the Markovian connection $\nabla^{(n)}$(cf. [10, 11]). Let

$$\varDelta_{O\left(M_\varepsilon^{(n)}\right)} := -\sum_{\alpha,i} B_{i,\alpha}B_{i,\alpha} + \sum_{\alpha,i} \delta^{(n)}\left(H_{i,\alpha}\right)\cdot B_{i,\alpha}.$$

Then $\varDelta_{O\left(M_\varepsilon^{(n)}\right)}$ is the lift of $L^{(n)}$ to the frame bundle.

Theorem (2.1): For any $f \in C^\infty\left(M_\varepsilon^{(n)}\right)$, we have

$$\varDelta_{O\left(M_\varepsilon^{(n)}\right)}(f \circ \pi) = \left(L^{(n)} f\right) \circ \pi.$$

Here π is the bundle projection.

Proof. It is a direct consequence of the identity $B_{i,\alpha}(f \circ \pi) = \left(H_{i,\alpha} f\right) \circ \pi$.

For any vector field Z on $M_\varepsilon^{(n)}$, we use F_Z to denote its scalarization, i.e. $F_Z(r) = r^{-1}(Z) \in H^{(n)}$.

Theorem (2.2): The following commutation relation holds:

$$\varDelta_{O\left(M_\varepsilon^{(n)}\right)} F_Z = F_{\Delta^{(n)}Z},$$

where

$$\Delta^{(n)} Z := \sum_{\alpha,i} \nabla_{h_{i,\alpha}}^{(n)} \nabla_{h_{i,\alpha}}^{(n)} Z + \delta^{(n)}\left(h_{i,\alpha}\right) \nabla_{h_{i,\alpha}}^{(n)} Z.$$

Proof. It is deduced from $B_{i,\alpha} F_Z = F_{\nabla_{h_{i,\alpha}}^{(n)} Z}$.

Now we define a Dirichlet form as follows:

$$\mathcal{E}_H^{(n)}(Z_1, Z_2) = E^{v_{n,\varepsilon}}\left(\sum_{\alpha,i} \langle \nabla_{h_{i,\alpha}}^{(n)} Z_1, \nabla_{h_{i,\alpha}}^{(n)} Z_2 \rangle_{M_\varepsilon^{(n)}}\right),$$
$$Z_1, Z_2 \in T\left(M_\varepsilon^{(n)}\right).$$

As a consequence of the relation

$$\mathrm{D}_{h_{i,\alpha}}^{(n)} \langle Z_1, Z_2 \rangle_{M_\varepsilon^{(n)}} = \langle \nabla_{h_{i,\alpha}}^{(n)} Z_1, Z_2 \rangle_{M_\varepsilon^{(n)}} + \langle Z_1, \nabla_{h_{i,\alpha}}^{(n)} Z_2 \rangle_{M_\varepsilon^{(n)}}$$

and integration by parts.

Proposition (2.3): For $s \in [0, \Delta\,]$,

$$\left\|V_{i+1}^{\mathcal{P}}(s)^{tr} V_{i+1}^{\mathcal{P}}(s) - (\Delta - s)^2 I\right\| \le 3(\Delta - s)^2\left(\cosh\left(2\sqrt{K_i^{\mathcal{P}}}\Delta\right)\cosh\left(8\sqrt{K_{i+1}^{\mathcal{P}}}\Delta\right) - 1\right), \quad (2.1)$$

$$\left\|S_i^{\mathcal{P}}(s)^{tr} S_i^{\mathcal{P}}(s) - s^2 I\right\| \le 3s^2\left(\cosh\left(2\sqrt{K_i^{\mathcal{P}}}\Delta\right) - 1\right) \quad (2.2)$$

and

$$\left\|V_{i+1}^{\mathcal{P}}(s)^{tr} S_{i+1}^{\mathcal{P}}(s) - (\Delta - s)sI\right\| \le 3s(\Delta - s)\left(\cosh\left(\sqrt{K_i^{\mathcal{P}}}\Delta\right)\cosh\left(5\sqrt{K_{i+1}^{\mathcal{P}}}\Delta\right) - 1\right). \quad (2.3)$$

Proof. We apply Lemma (6.2.19)[12]. For operators A and B and real numbers a and b,

$$A^{tr}B - abI = (A^{tr} - aI)(B - bI) + a(B - bI) + b(A^{tr} - aI).$$

The asserted inequalities now follow with judicious choices for A and B as well as Eqs. (3.6) and (3. 7) along with the fact that $s/\Delta \le 1$.

Applying Proposition (2.3) to Eqs (6.127)[12] and (6.128)[12] gives the estimates we need on $\mathcal{R}_{\mathcal{P}}$ to continue forward.

Proposition (2.4): Let $Y_{\mathcal{P}}$ be as in Eq. (6.177) [12], $\tau_{\mathcal{P}}$ be as in Eq. (6.178) [12], and τ_G be as in Eq. (6.180)[12]. There is a constant C = C (d, curvature) < ∞ such that,

$$\int_{H_{\mathcal{P}}^{\varepsilon}(\mathbb{R}^d)} \left|e^{Y_{\mathcal{P}}(\omega)} - e^{-\tau_G \int_0^1 Scal(\phi(\omega)(s))}\right| d\mu_{S_{\mathcal{P}}}(\omega) \le C\left(\sqrt{|\tau_{\mathcal{P}} - \tau_G|} + \Delta^{1/4}\right). \quad (2.4)$$

Proof. Breaking the integrand into pieces we consider

$$\underbrace{e^{Y_{\mathcal{P}}(\omega)} - e^{-\tau_G \mathcal{R}_{\mathcal{P}}}}_{I} + \underbrace{e^{-\tau_G \mathcal{R}_{\mathcal{P}}} - e^{-\tau_G S_{\mathcal{P}}}}_{II} + \underbrace{e^{-\tau_G S_{\mathcal{P}}} - e^{-\tau_G \int_0^1 Scal(\phi(\omega)(s))}}_{III}.$$

Let $\Lambda = \Lambda$ (curvature) $< \infty$ be given such that $|Scal| \leq \Lambda$. Then,

$$\left|e^{-\tau_G \mathcal{R}_{\mathcal{P}}} - e^{-\tau_G S_{\mathcal{P}}}\right| \leq e^{\tau_G \Lambda}\left|e^{-\tau_G(\mathcal{R}_{\mathcal{P}} - S_{\mathcal{P}})} - 1\right|.$$

Now applying Lemma (6.2.48)[12] and Lemma (6.2.5)[12],

$$\int_{H_{\mathcal{P}}^{\varepsilon}(\mathbb{R}^d)} |II| \leq e^{\tau_G \Lambda} \int_{H_{\mathcal{P}}(\mathbb{R}^d)} \left|e^{-\tau_G(\mathcal{R}_{\mathcal{P}} - S_{\mathcal{P}})} - 1\right| \leq C(e^{C\Delta} - 1)^{1/2}.$$

Similarly, with

$$\left|e^{-\tau_G S_{\mathcal{P}}} - e^{-\tau_G \int_0^1 Scal(\phi(\omega)(s))}\right| \leq e^{\tau_G \Lambda}\left(exp\left\{\tau_G \left|S_{\mathcal{P}} - \int_0^1 Scal\big(\phi(\omega)(s)\big)\, ds\right|\right\} - 1\right),$$

another application of Lemma (6.2.48) [12] to the right-hand side along with Lemma (6.2.6) [12] gives $\int |III| \leq C\Delta^{1/4}$.

What remains then is to bound $\int |I|$. To start, we will assume that Λ is also a bound on Ric so that $\left|\langle Ric_{u(s_{i-1})}\Delta_i b, \Delta_i b\rangle\right| \leq \Lambda \|\Delta_i b\|^2$ for each $i = 1, 2, \ldots, n$. From here,

$$\tau_G \mathcal{R}_{\mathcal{P}} + Y_{\mathcal{P}} = (\tau_G - \tau_{\mathcal{P}})\mathcal{R}_{\mathcal{P}} + \tau_{\mathcal{P}} \partial \mathcal{R}_{\mathcal{P}}$$

where

$$\partial \mathcal{R}_{\mathcal{P}} := \langle Ric_{u(s_0)}\Delta_1 b, \Delta_1 b\rangle + \langle Ric_{u(s_{n-2})}\Delta_{n-1} b, \Delta_{n-1} b\rangle + \langle Ric_{u(s_{n-1})}\Delta_n b, \Delta_n b\rangle.$$

Using the bounds

$$|\mathcal{R}_{\mathcal{P}}| \leq \Lambda \sum_{i=1}^{n} \|\Delta_i b\|^2 \;\; and \;\; |\partial \mathcal{R}_{\mathcal{P}}| \leq \Lambda(\|\Delta_1 b\|^2 + \|\Delta_{n-1} b\|^2 + \|\Delta_n b\|^2)$$

along with Eq. (3. 4), Eq. (6.200) in Lemma (6.2.46)[12], and Theorem (6.2.4)[12] we have

$$\int_{H_{\mathcal{P}}^{\varepsilon}(\mathbb{R}^d)} e^{2\tau_G |\mathcal{R}_{\mathcal{P}}|} d\mu_{S_{\mathcal{P}}} \leq 2\tau_G \Lambda \sum_{i=1}^{n} \mathbb{E}\left[\|\Delta_i b\|^2 e^{2\tau_G \Lambda \sum_{j=1}^{n} \|\Delta_j b\|^2}\right] + 1 \leq C. \tag{2.5}$$

Along these same lines, from Eq. (6.200)[12],

$$\int_{H_{\mathcal{P}}^{\varepsilon}(\mathbb{R}^d)} |(\tau_G - \tau_{\mathcal{P}})\mathcal{R}_{\mathcal{P}}| e^{|(\tau_G - \tau_{\mathcal{P}})\mathcal{R}_{\mathcal{P}} + \tau_{\mathcal{P}} \partial \mathcal{R}_{\mathcal{P}}|} d\mu_{S_{\mathcal{P}}} \leq \mathbb{E}\left[|(\tau_G - \tau_{\mathcal{P}})\mathcal{R}_{\mathcal{P}}| e^{|(\tau_G - \tau_{\mathcal{P}})\mathcal{R}_{\mathcal{P}} + \tau_{\mathcal{P}} \partial \mathcal{R}_{\mathcal{P}}|}\right]$$

$$\leq |\tau_G - \tau_{\mathcal{P}}| \Lambda \sum_{i=1}^{n} \mathbb{E}\left[\|\Delta_i b\|^2 e^{2\Lambda \sum_{j=1}^{n} \|\Delta_j b\|^2}\right] \leq C(|\tau_G - \tau_{\mathcal{P}}|),$$

and arguing similarly using Eq. (6.199)[12],

$$\int_{H_{\mathcal{P}}^{\varepsilon}(\mathbb{R}^d)} |\tau_{\mathcal{P}} \partial \mathcal{R}_{\mathcal{P}}| e^{|(\tau_G - \tau_{\mathcal{P}})\mathcal{R}_{\mathcal{P}} + \tau_{\mathcal{P}} \partial \mathcal{R}_{\mathcal{P}}|} d\mu_{S_{\mathcal{P}}} \leq \mathbb{E}\left[|\tau_{\mathcal{P}} \partial \mathcal{R}_{\mathcal{P}}| e^{|(\tau_G - \tau_{\mathcal{P}})\mathcal{R}_{\mathcal{P}} + \tau_{\mathcal{P}} \partial \mathcal{R}_{\mathcal{P}}|}\right] \leq C\Delta.$$

In particular,

$$\mathbb{E}\left[|(\tau_G - \tau_{\mathcal{P}})\mathcal{R}_{\mathcal{P}} + \tau_{\mathcal{P}} \partial \mathcal{R}_{\mathcal{P}}| e^{|(\tau_G - \tau_{\mathcal{P}})\mathcal{R}_{\mathcal{P}} + \tau_{\mathcal{P}} \partial \mathcal{R}_{\mathcal{P}}|}\right] \leq C(|\tau_G - \tau_{\mathcal{P}}| + \Delta). \tag{2.6}$$

With Eqs. (6.182)[12] and (3. 4), Lemma (6.2.48)[12] implies $\int |I| \leq C(|\tau_G - \tau_{\mathcal{P}}| + \Delta)^{1/2}$. Combining the bounds on $\int |I|, \int |II|$, and $\int |III|$ concludes the proof.

III. Claims

Definition (3.1): On $M_\varepsilon^{(n)}$ a Riemannian metric is defined by the condition that $\Theta_v^{(n)}$ is an isometry of $T_v\left(M_\varepsilon^{(n)}\right)$ onto $H^{(n)}$.

Hereafter, we shall use $\langle\cdot,\cdot\rangle_{M_\varepsilon^{(n)}}$ or simply $\langle\cdot,\cdot\rangle$ to denote this metric.

Note that under the map π_n^W, I_n, we can identify $M_\varepsilon^{(n)}, H_\varepsilon^{(n)}(M)$ and $H_\varepsilon^{(n)}$. In particular, we have

$$d(v_i, v_{i+1}) = \int_{s_i}^{s_{i+1}} |\dot\sigma_v(s)|ds = \|\Delta_i b_v\|,$$

where $\dot b_v(s) = t_{0\leftarrow s}^{\sigma_v}\dot\sigma_v(s)$. It is sometimes convenient not to distinguish $M_\varepsilon^{(n)}$ and $H_\varepsilon^{(n)}(M)$.

We define the following Markovian connection which is compatible with $\langle\cdot,\cdot\rangle_{M_\varepsilon^{(n)}}$ on the finite manifold $M_\varepsilon^{(n)}$

Theorem (3.2): For every vector field $z: M_\varepsilon^{(n)} \longmapsto H^{(n)}$ such that $z^\alpha(s_i) \in C_r^1(M_\varepsilon^{(n)})$ for $\alpha = 1, \dots, d,\ i = 1, \dots, n$, we have

$$\int_{M_\varepsilon^{(n)}} D_z^{(n)} f \, dv_{n,\varepsilon} = \int_{M_\varepsilon^{(n)}} f \cdot \delta^{(n)} z \, dv_{n,\varepsilon},$$

where

$$(\delta^{(n)} z)(\sigma) = \sum_\alpha \int_0^1 \dot z^\alpha(t^-) d[I_n^{-1}(\sigma)]^\alpha(t) - \int_0^1 D_{t,\alpha}^X \dot{\hat z}^\alpha(t^-)dt - \frac{2D_z^{(n)}\hat\varphi_n}{\hat\varphi_n} \qquad (3.1)$$

and $\hat z$ is given in (6.9)[1].

Proof. The idea of the proof is to push back all divergence computations to the flat finite dimensional Gaussian vector space $H^{(n)}$.

For any $f \in C_r^\infty(M_\varepsilon^{(n)})$, we have

$$\int_{M_\varepsilon^{(n)}} D_z^{(n)} f \, dv_{n,\varepsilon} = \int_{H_\varepsilon^{(n)}(M)} D_z^{(n)} f \cdot \hat\varphi_n^2 dv_n = \int_{H_\varepsilon^{(n)}(M)} D_z^{(n)}(f \cdot \hat\varphi_n^2) dv_n - \int_{H_\varepsilon^{(n)}(M)} f D_z^{(n)} \hat\varphi_n^2 dv_n.$$

The function $(f \cdot \hat\varphi_n^2) \circ I_n$ belongs to $C^\infty(H^{(n)})$. We define

$\phi_n \in C^\infty(H^{(n)})$ satisfying

$$\begin{cases} \varphi_n(b) = 1, & b \in H_\varepsilon^{(n)}, \\ \varphi_n(b) = 0, & b \notin H_{\varepsilon''}^{(n)}, \end{cases} \qquad (3.2)$$

where $\varepsilon' < \varepsilon < \varepsilon''$, $H_{\varepsilon'}^{(n)} \subset H_\varepsilon^{(n)} \subset H_{\varepsilon''}^{(n)}$. Then $\phi_n \cdot \hat z$ is a vector field on $H^{(n)}$. By the intertwinning formula (6.7)[1], we have

$$\begin{aligned}
&\int_{H_\varepsilon^{(n)}(M)} D_z^{(n)}(f \cdot \hat\varphi_n^2) dv_n \\
&= \int_{H_\varepsilon^{(n)}} D_{\hat z}^X\big((f \cdot \hat\varphi_n^2) \circ I_n\big) d\mu_n \\
&= \int_{H^{(n)}} D_{\phi_n \cdot \hat z}^X\big((f \cdot \hat\varphi_n^2) \circ I_n\big)\, d\mu_n \\
&= \int_X \big[D_{\phi_n \cdot \hat z}^X\big((f \cdot \hat\varphi_n^2) \circ I_n\big)\big] \circ \pi_n^X \, d\mu \\
&= \int_X [(f \cdot \hat\varphi_n^2) \circ I_n \circ \pi_n^X] \delta(\phi_n \cdot \hat z)\, d\mu.
\end{aligned}$$

Consider the following formula (cf. [9]):

$$\int_0^1 f_\alpha(s)\, \dot b_n^\alpha(s)\, ds = \int_0^1 \left(\frac{1}{\Delta s}\int_{s^-}^{s^+} f_\alpha(t)\, dt\right) db^\alpha(s) + \sum_\alpha \int_0^1 \left(\frac{1}{\Delta s}\int_{s^-}^{s^+} D_{s,\alpha}^X f_\alpha(t)\, dt\right) ds \qquad (3.3)$$

where $f(s)$ is a non-adapted $\mathbb{R}^d$ valued process with some regularity assumptions and the stochastic integral is taken in the sense of Skorohod.

Applying this formula to $f_\alpha(s) = \phi_n \dot{\hat z}^\alpha(s^-)$,

$$\begin{aligned}
\delta(\phi_n \cdot \hat z)(b) &= \sum_\alpha \int_0^1 \phi_n \cdot \dot z^\alpha(t^-) \dot b_n^\alpha(t)\, dt - \int_0^1 D_{t,\alpha}^X \left[\phi_n \dot{\hat z}^\alpha(t^-)\right) dt = \sum_\alpha \int_0^1 \phi_n \cdot \dot z^\alpha(t^-) \dot b_n^\alpha(t)\, dt \\
&\quad - \sum_\alpha \int_0^1 \phi_n \cdot \left(\int_{t^-}^{t^+} \frac{t^+ - s}{t^+ - t^-} \Omega_{\gamma\lambda\beta}^\alpha(r_n(s)) \dot b_n^\gamma(s) \bar z^\lambda(s) \dot b_n^\beta(s)\, ds\right) \dot b_n^\alpha(t)\, dt \\
&\quad - \sum_\alpha \int_0^1 \phi_n \cdot \left(\int_0^{t^-} \Omega_{\gamma\lambda\beta}^\alpha(r_n(s)) \dot b_n^\gamma(s) \bar z^\lambda(s)\, ds\right) \cdot \dot b_n^\beta(t) \dot b_n^\alpha(t)\, dt \\
&\quad - \int_0^1 \phi_n D_{t,\alpha}^X \dot{\hat z}^\alpha(t^-)\, dt - \int_0^1 \dot{\hat z}^\alpha(t^-) D_{t,\alpha}^X \phi_n\, dt,
\end{aligned}$$

where $b_n = \pi_n^X(b)$ and r_n is the horizontal lift of $I_n(b_n)$ satisfying

$$dr_n(s) = \sum_{\alpha=1}^d A_\alpha(r_n) \dot b_n^\alpha(s)\, ds, \quad r_n(0) = r_0.$$

In view of the antisymmetry of $\Omega^{\alpha}_{\gamma\lambda\beta}$ the second and third terms of this expression vanish. By the construction of ϕ_n, we know that $\varphi_n \cdot D^X_{t,\alpha}\phi_n = 0$. Hence

$$\int_{H_\varepsilon^{(n)}(M)} D_z^{(n)}(f\cdot\hat{\varphi}_n^2)dv_n = \int_{H_\varepsilon^{(n)}}(f\cdot\hat{\varphi}_n^2)\circ I_n(b)\cdot\left[\sum_\alpha\int_0^1 \dot{z}^\alpha(t^-)\dot{b}_n^\alpha(t)\,dt - \int_0^1 D^X_{t,\alpha}\dot{z}^\alpha(t^-)\,dt\right]d\mu_n(b)$$

$$= \int_{H_\varepsilon^{(n)}(M)}(f\cdot\hat{\varphi}_n^2)(\sigma)\cdot\left[\sum_\alpha\int_0^1 \dot{z}^\alpha(t^-)d[I_n^{-1}(\sigma)]^\alpha(t) - \int_0^1 D^X_{t,\alpha}\dot{z}^\alpha(t^-)\,dt\right]dv_n(\sigma)$$

$$= \int_{M_\varepsilon^{(n)}} f(\sigma)\cdot\left[\sum_\alpha\int_0^1 \dot{z}^\alpha(t^-)d[I_n^{-1}(\sigma)]^\alpha(t) - \int_0^1 D^X_{t,\alpha}\dot{z}^\alpha(t^-)\,dt\right]dv_{n,\varepsilon}(\sigma).$$

Finally, we obtain

$$\int_{M_\varepsilon^{(n)}} D_z^{(n)}f\,dv_{n,\varepsilon} = \int_{M_\varepsilon^{(n)}} f\delta^{(n)}z\,dv_{n,\varepsilon},$$

where $\delta^{(n)}z$ is given by (6.12)[1].

Theorem (3.3): For any $g\in C_b(W(M))$, let g_n be the projection of g (see Definition (6.1.17)). Then for any t > 0, we have

$$\tilde{T}_t^{(n)}g_n \xrightarrow{w\cdot} T_t g \quad in \quad L^2(W(M),v),$$

where $T_t := e^{-tL}$.

Proof. Following the notation of the previous theorem's proof, for any $f\in E_1$, we have

$$E^v(\tilde{T}_t^{(n)}g_n\cdot f) - E^v(T_t g\cdot f) \le E^v(\tilde{T}_t^{(n)}g_n\cdot\tilde{f}_n) - E^v(T_t g\cdot f) + E^v\left(\tilde{T}_t^{(n)}g_n\cdot(f-\tilde{f}_n)\right),$$

where f_n is the projection of f. The second term clearly converges to zero. By Theorem (6.1.29) the first is equal to

$$E^{v_{n,\varepsilon}}(T_t^{(n)}g_n\cdot f_n) - E^v(T_t g\cdot f) = E^P\big(g_n(p_t^n)f_n(p_0^n)\big) - E^P\big(g(p_t)\cdot f(p_0)\big)$$
$$= E^P\big([g_n(p_t^n)-g(p_t^n)]f_n(p_0^n)\big) + E^P\big(g(p_t^n)[f_n(p_0^n)-f(p_0^n)]\big)$$
$$+E^P\big(g(p_t^n)[f(p_0^n)-f(p_0)]\big) + E^P\big([g(p_t^n)-g(p_t)]\cdot f(p_0)\big).$$

The convergence of the last two terms follows from the a.s. convergence of p_t^n to p_t and the dominated convergence theorem. As for the first, since $v_{n,\varepsilon}$ is the invariant measure of p_t^n, it is estimated by

$$CE^P|g_n(p_t^n)-g(p_t^n)| = CE^{v_{n,\varepsilon}}|g_n(\sigma_v)-g(\sigma_v)|$$
$$= CE^\mu(\varphi_n^2|g_n\circ I_n\circ\pi_n^X - g\circ I_n\circ\pi_n^X|)$$
$$\le CE^\mu(\varphi_n^2|g\circ I - g\circ I_n\circ\pi_n^X|)\to 0.$$

The second is similar.

Proposition (3.4): Under the assumptions of Theorem (6.2.1), the limit in Eq. (6.56) is zero.

Proof. Combining Propositions (6.2.35), (6.2.39), (6.2.40), and Eqs. (6.179) and (6.180) shows that the limit in Eq. (6.55) vanishes when Δ→ 0.

Here we collect several inequalities which are straight forward to show, but the frequency of use warrants their mention. For any $a\in\mathbb{R}$ and $p\in\mathbb{N}$,

$$|e^a-1|^p \le e^{p|a|}-1 \le p|a|e^{p|a|}. \qquad (3.4)$$

If $a,b>0$ and $\alpha\ge 1$,

$$\frac{\sinh(a)}{a} \le \cosh(a), \qquad (3.5)$$

$$\cosh(a)\cosh(b) \le \cosh(a+b), \qquad (3.6)$$

$$\cosh(a)\,(\cosh(b)-1) \le \cosh(a)\cosh(b)-1, \qquad (3.7)$$

$$\alpha(\cosh(a)\cosh(b)-1) \le \cosh(\alpha a)\cosh(\alpha b)-1. \qquad (3.8)$$

References:

1. Cruzeiro, A.B., & Zhangb, X. (2003). Finite dimensional approximation of Riemannian path space geometry. *Journal of Functional Analysis*, 205 (2003), 206–270.
2. Cruzeiro, A.B., & Malliavin, P. (1996). Renormalized differential geometry on path space: structural equation, curvature, *J. Funct. Anal*., 139 (1996), 119–181.
3. Cruzeiro, A.B., & Malliavin, P. (2000). Frame bundle of Riemannian path space and Ricci tensor in adapted differential geometry, *J. Funct. Anal.*, 177 (2000), 219–253.
4. Andersson, L., & Driver, B. (1999). Finite dimensional approximation to Wiener measure and path integral formulas on manifolds, *J. Funct. Anal.*, 165 (1999), 430–498.
5. Eberle, A. (1997). Diffusions on path and loop spaces: existence, finite dimensional approximation and Hölder continuity, *Probab. Theory Related Fields*, 109 (1997), 77–99.

6. Röckner, M., & Zhang, T.S. (1996). Finite dimensional approximation of diffusion processes on infinite dimensional spaces, *Stoch. Stoch. Rep.*, 57 (1996), 37–55.
7. Fang, S., & Malliavin, P. (1993). Stochastic Analysis on the path space of a Riemannian manifold: I. Markovian Stochastic Calculus, *J. Funct. Anal.*, 118 (1993), 249–274.
8. Kobayashi, N. (1962). *Differential Geometry*, Wiley, New York.
9. (1997). Shigekawa, L^p-Contraction Semigroup for Vector Valued Functions, *J. Funct. Anal.*, 147 (1997), 69–108.
10. Laetsch, T. (2013). An approximation to Wiener measure and quantization of the Hamiltonian on manifolds with non-positive sectional curvature, *Journal of Functional Analysis*, 265 (2013), 1667–1727.
11. Andersson, L., & Driver, B.K. (1999). Finite dimensional approximations to wiener measure and path integral formulas on manifolds, *J. Funct. Anal.*, 165 (1999), 430–498.
12. Shigeki, A. (2007). Semi-classical limit of the bottom of spectrum of a Schrödinger operator on a path space over a compact Riemannian manifold, *J. Funct. Anal.*, 251 (1) (2007), 59–121.
13. Bär, C., & Pfäffle, F. (2008). Path integrals on manifolds by finite dimensional approximation, *J. Reine Angew. Math*., 625 (2008), 29–57.
14. Butko, Ya.A. (2008). Feynman formulas and functional integrals for diffusion with drift in a domain on a manifold, *Mat. Zametki*, 83 (3) (2008), 333–349.
15. Cruzeiro, A.B., & Zhang, X. (2003). Finite dimensional approximation of Riemannian path space geometry, *J. Funct. Anal.* 205 (1) (2003) 206–270.
16. Durhuus, B., & Jonsson, T. (2001). Discrete approximations to integrals over unparametrized paths, *J. Math. Phys.*, 42 (8) (2001), 3769–3788.
17. Inahama, Y. (2005). On the piecewise geodesic approximation of Andersson and Driver, *Osaka J. Math*., 42 (4) (2005), 791–806.
18. Obrezkov, O.O. (2003). The proof of the Feynman–Kac formula for heat equation on a compact Riemannian manifold, *Infin. Dimens. Anal. Quantum Probab. Relat. Top*., 6 (2) (2003), 311–320.
19. Smolyanov, O.G., Weizsäcker, H.V., & Wittich, O. (2007). Chernoff's theorem and discrete time approximations of Brownian motion on manifolds, *Potential Anal*., 26 (1) (2007), 1–29.
20. Telyatnikov, I.V. (2008). Smolyanov–Weizsäcker surface measures generated by diffusions on the set of trajectories in Riemannian manifolds, *Infin. Dimens. Anal. Quantum Probab. Relat. Top*., 11 (1) (2008), 21–31.
21. Exner, P., Neidhardt, H., & Zagrebnov, V. A. (2011). Remarks on the Trotter–Kato product formula for unitary groups, *Integral Equations Operator Theory*, 69 (4) (2011), 451–478.
22. Lim, A.P.C. (2007). Path integrals on a compact manifold with non-negative curvature, *Rev. Math. Phys*., 19 (2007), 967–1044.
23. Driver, B.K. (2004). Curved Wiener space analysis, in: Real and Stochastic Analysis, in: Trends Math., Birkhäuser Boston, (pp.43-198). Boston, MA.
24. Hsu, E.P. (2002). Stochastic Analysis on Manifolds, *Grad. Stud. Math*., vol. 38, American Mathematical Society, Providence, RI.
25. Butko, Ya.A. (2004). Representations of the solution of the Cauchy–Dirichlet problem for the heat equation in a domain of a compact Riemannian manifold by functional integrals, *Russ. J. Math. Phys*., 11 (2) (2004), 121–129.
26. Butko, Ya.A. (2006). Functional integrals corresponding to a solution of the Cauchy–Dirichlet problem for the heat equation in a domain of a Riemannian manifold, *Fundam. Prikl. Mat.*, 12 (6) (2006), 3–15.
27. Mümken, B. (2008). A dynamical zeta function for pseudo Riemannian foliations, *Comm. Math. Phys*., 279 (3) (2008), 585–594.