SOI: 1.1/TAS DOI: 10.15863/TAS
International Scientific Journal
Theoretical & Applied Science

p-ISSN: 2308-4944 (print) **e-ISSN:** 2409-0085 (online)

Year: 2021 **Issue:** 10 **Volume:** 102

Published: 11.10.2021 http://T-Science.org

QR – Issue QR – Article

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ABOUT NON-ARCHIMEDIAN FUNCTION DYNAMICAL SYSTEM

***Abstract**: In this paper we consider the discrete time p -adic dynamic system of the family of rational functions in the form $\frac{1}{x^2+a}$. In order to solve the problem in this study, a number of real non-negative functions were constructed using the properties of the p -adic norm and some substitutions.The following conclusions were drawn about the discrete time dynamics of p-adic rational functions under consideration using their results by studying their dynamics:*

This rational function cannot have a unique fixed point, the parameter a has two fixed points at a single value of $a = -\frac{3}{\sqrt[3]{4}}$, and the parameter a has three fixed points at the values of $a \neq -\frac{3}{\sqrt[3]{4}}$ proved to be. The p -adic dynamical system with two fixed points was studied at $p = 2$. Conditions were found for the parameters that attractor and indifferent fixed points. Also, basin of attraction, Siegel disks were found and trajectories were studied.

***Key words**: p -adic norm,fixed point, attractor fixed point, basin of attraction, indifferent fixed point, Siegel disk, a maximum Siegel disk (SI((x)), 2-adic norm, open ball, closed ball, sphere.*

***Language**: English*

***Citation**: Abdukaxorova, Z. T. (2021). About non-Archimedian function dynamical system. *ISJ Theoretical & Applied Science, 10 (102)*, 413-418.

***Soi**: http://s-o-i.org/1.1/TAS-10-102-30 ***Doi**: crossref https://dx.doi.org/10.15863/TAS.2021.10.102.30

***Scopus ASCC**: 2600.*

Introduction

In the world many scientific and applied research are reduced to the studies that have focused on discrete-time dynamics of the functions defined in Archimedean or non-Archimedean fields. p-Adic dynamical systems generated by rational functions are effective in informatics, digital analysis and cryptography, psychodynamics and automation theory, genetic coding and population management. In p-adic analysis, rational functions play an important role similar to those of analytical functions in complex analysis. Therefore, the study of the dynamics of rational functions in the field of p-adic numbers is one of the most important tasks in the theory of dynamical systems.

It is known that the analytic functions play important role in complex analysis. In the p -adic analysis the rational functions play a similar role to the analytic functions in complex analysis [1]. Therefore, naturally one arises a question to study the dynamics of these functions in the p -adic analysis.

The study of p -adic dynamical systems arises in Diophantine geometry in the constructions of canonical heights, used for counting rational points on algebraic vertices over a number field, as in [2].

In [3, 4] p -adic field have arisen in physics in the theory of superstrings, promoting questions about their dynamics. Also some applications of p -adic dynamical systems to some biological, physical systems has been proposed in [5,7,8,3,9].

Moreover p -adic dynamical systems are effective in computer science (straight line programs), in numerical analysis and in simulations (pseudorandom numbers), uniform distribution of sequences, cryptography (stream ciphers, T -functions), combinatory (Latin squares), automata theory and formal languages, genetics. The monograph [10] contains the corresponding survey (see also [11,12] for the theory and applications of p -adic dynamical systems).

In [7, 9] the behavior of a p -adic dynamical system $f(x) = x^n$ in the fields of p -adic numbers $\mathbb{Q}_p$ and $\mathbb{C}_p$ were studied.

In [6] the properties of the nonlinear p -adic dynamic system $f(x) = x^2 + c$ with a single parameter c on the integer p -adic numbers $\mathbb{Z}_p$ are investigated. This dynamic system illustrates possible brain behaviors during remembering.

In [13], dynamical systems defined by the functions $f_q(x) = x^n + q(x)$, where the perturbation $q(x)$ is a polynomial whose coefficients have small p -adic absolute value, was studied.

Preliminaries

***p* -adic numbers**. Let $\mathbb{Q}$ be the field of rational numbers and p is a fixed prime number. The greatest common divisor of the positive integers n and m is denoted by (n, m). Every rational number $x \neq 0$ can be represented in the form $x = p^{\gamma(x)}\frac{n}{m}$, where $\gamma(x), n\epsilon\mathbb{Z}$, m is a positive integer, $(p, n) = 1, (p, m) = 1$.

The p *-adic norm* of rational number $x$is given by

$$|x|_p = \begin{cases} p^{-\gamma(x)}, \text{for} x \neq 0, \\ 0, \quad \text{for} x = 0. \end{cases}$$

It has following properties:

1) $|x|_p \geq 0$ and $|x|_p = 0$ if and only if $x = 0$.

2) $|xy|_p = |x|_p|y|_p$,

3) The strong triangle inequality $|x + y|_p \leq \max\{|x|_p, |y|_p\}$,

3.1) if $|x|_p \neq |y|_p$ then $|x + y|_p = \max\{|x|_p, |y|_p\}$

3.2) if $|x|_p = |y|_p$ then $|x + y|_p \leq |x|_p$,

This is a non-Atchimedean one.

The completion of $\mathbb{Q}$ with respect to p -adic norm defines the p -adic $\mathbb{Q}_p$.

The algebraic completion of $\mathbb{Q}_p$ is denoted by $\mathbb{C}_p$ and it is called *complex p-adic numbers*. For any $a \in \mathbb{C}_p$ and $r > 0$ denote

$$U_r(a) = \{x \in \mathbb{C}_p : |x - a|_p < r\},$$
$$V_r(a) = \{x \in \mathbb{C}_p : |x - a|_p \leq r\},$$
$$S_r(a) = \{x \in \mathbb{C}_p : |x - a|_p = r\}.$$

Dynamical system in $\mathbb{C}_p$. Recall some known facts concerning dynamical systems (f, U) in $\mathbb{C}_p$, where $f: U \to f(x) \in U$ is an analytic function and $U = U_r(a)$ or $\mathbb{C}_p$.

Now let $f: U \to U$ be an analytic function. Denote $f^n = \underbrace{f \circ \cdots \circ f}_{n}$.

If $f(x_0) = x_0$ then x_0 is called a fixed point. The set of all fixed points of f is denoted by $Fix(f)$. A fixed point x_0 is called an attractor if there exists a neighborhood $U(x_0)$ of x_0 such that for all points $x \in U(x_0)$ it holds $\lim_{n\to\infty} f^n(x) = x_0$. If x_0 is an attractor then its basin of attraction is

$$A(x_0) = \{x \in \mathbb{C}_p : f^n(x) \to x_0, n \to \infty\}.$$

Let x_0 be a fixed point of a function $f(x)$. Put $\lambda = f'(x_0)$. The point x_0 is attractive if $0 < |\lambda|_p < 1$, indifferent if $|\lambda|_p = 1$.

The ball $U_r(x_0)$ (contained in V) is said to be a Siegel disk if each sphere $S_\rho(x_0), \rho < r$ is an invariant sphere of $f(x)$, i.e. if $x\epsilon S_\rho(x_0)$ then all iterated points $f^n(x)\epsilon S_\rho(x_0)$ for all $n = 1,2, ...$ The union of all Siegel disks with the center at x_0 is said to a maximum Siegel disk and denoted by $SI(x_0)$.

Main part

In this paper we considered the function f can be written in the following form:

$$f(x) = \frac{1}{x^2+a}, \quad a \in \mathbb{C}_p, \qquad (1)$$

where $x \neq \hat{x}_{1,2} = \pm\sqrt{-a}$.

It is easy to see that for rational function (1) the equation $f(x) = x$ for fixed points is equivalent to the equation

$$x^3 + ax - 1 = 0. \qquad (2)$$

Since $\mathbb{C}_p$ is algebraic closed the equation (2) may have three solution with one of the following relations:

(i) One solution having multiplicity three;

(ii) Two solutions, one of which has multiplicity two;

(iii) Three distinct solutions.

Theorem 1. For (1) rational functions, the following holds:

1. (1) rational function cannot have a unique fixed point.

2. The function (1) has two distinct fixed points if and only if $a = -\frac{3}{\sqrt[3]{4}}$.

Proof. 1. Assume (1) has a unique fixed point, say x_0. Then the LHS of equation (2) (which is equivalent to $f(x) = x$) can be written as

$$x^3 + ax - 1 = x^3 - 3x_0x^2 + 3x_0^2x - x_0^3.$$

Consequently,

$$\begin{cases} -3x_0 = 0 \\ 3x_0^2 = a \\ x_0^3 = 1 \end{cases}.$$

It is easy to see from the last equations that our assume is incorrect. Hence, (1) function does not have a unique fixed point.

2. Denote by x_1 and x_2 solution of equation (2), x_1 has multiplicity two. Then we have $x^3 + ax - 1 = (x - x_1)^2(x - x_2)$ and

$$x^3 + ax - 1 = x^3 - (2x_1 + x_2)x^2 + (2x_1x_2 + x_1^2)x - x_1^2x_2.$$

Hence,

$$\begin{cases} 2x_1 + x_2 = 0 \\ 2x_1x_2 + x_1^2 = a. \\ x_1^2x_2 = 1 \end{cases}$$

As are result

$$\begin{cases} x_1 = -\frac{1}{\sqrt[3]{2}} \\ x_2 = \sqrt[3]{4} \\ a = -\frac{3}{\sqrt[3]{4}} \end{cases}.$$

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	JIF = 1.500	SJIF (Morocco) = 7.184	OAJI (USA) = 0.350

The function has $x_1 = -\frac{1}{\sqrt[3]{2}}$ and $x_2 = \sqrt[3]{4}$ two fixed points at a single value of $a = -\frac{3}{\sqrt[3]{4}}$. Theorem is proved.

Corollary. If $a \neq -\frac{3}{\sqrt[3]{4}}$, then the function (1) has three distinct fixed points.

We know the rational function (1) has two distinct fixed points if and only if $a = -\frac{3}{\sqrt[3]{4}}$. When $a = -\frac{3}{\sqrt[3]{4}}$, it is easy to see that (1) function has two distinct fixed points $x_1 = -\frac{1}{\sqrt[3]{2}}$ and $x_2 = \sqrt[3]{4}$

Let $f: U \to U$ and $g: V \to V$ be two maps. $f$and g are said to be topologically conjugate if there exists a homeomorphism $h: U \to V$ such that, $h \circ f = h \circ g$. The homeomorphism h is called a topological conjugacy. Mappings that are topologically conjugate are completely equivalent in terms of their dynamics. For example, if f is topologically conjugate to g via h, and x_0 is a fixed point for f, then$h(x_0)$ is fixed point for g. Indeed, $h(x_0) = hf(x_0) = gh(x_0)$.

Let homeomorphism $h: \mathbb{C}_p \to \mathbb{C}_p$ is defined by $x = h(t) = t + x_1 = t - \frac{1}{\sqrt[3]{2}}$. So $h^{-1}(x) = x + \frac{1}{\sqrt[3]{2}}$. Note that, the function f is topologically conjugate $h^{-1} \circ f \circ h$. We have

$$f(x) = \frac{\frac{1}{\sqrt[3]{2}}x^2 - \sqrt[3]{2}x}{x^2 - \sqrt[3]{4}x - \sqrt[3]{2}}, \qquad (3)$$

where $x \neq \breve{x}_{1,2} = \frac{1\pm\sqrt{3}}{\sqrt[3]{2}}$.

Thus we study the dynamical system $(f, \mathbb{C}_p)$ with f given by (3).Note that, function (3) has two fixed points $x_1 = 0$ and $x_2 = \frac{3}{\sqrt[3]{2}}$. So we have $f'(x_1) = 1$ and $f'(x_2) = 8$. Thus, the point $x_1 = 0$ is an indifferent point for (3). For any $x \in \mathbb{C}_p$, $x \neq \breve{x}_{1,2}$, by simple calculation we get

$$|f(x)|_p = |x|_p \frac{\left|\frac{1}{\sqrt[3]{2}}x - \sqrt[3]{2}\right|_p}{|x-\breve{x}_1|_p |x-\breve{x}_2|_p}. \qquad (4)$$

Denote $P = \{x \in \mathbb{C}_p : \exists n \in \mathbb{N} \cup \{0\},\ f^n(x) \in \{\breve{x}_1, \breve{x}_2\}\}$.

Case $p = 2$.

Now let us calculate the 2-adic norm of $\breve{x}_1$ and $\breve{x}_2$. We know $\sqrt{3} \notin \mathbb{Q}_2$. Consider the quadratic extension of $K = \mathbb{Q}_2(\sqrt{3})$. We can write any element of Kin the form $a + b\sqrt{3}$. $\mathbb{N}_{K\backslash\mathbb{Q}_2}(a + b\sqrt{3}) = a^2 - 3b^2$.

$$|1+\sqrt{3}|_2 = \sqrt{\left|\mathbb{N}_{K\backslash\mathbb{Q}_2}(1+\sqrt{3})\right|_2} = \sqrt{|1-3|_2} = \frac{1}{\sqrt{2}}.$$

We know $\sqrt[3]{2} \notin \mathbb{Q}_2$. Consider the cubic extension of $K = \mathbb{Q}_2(\sqrt[3]{2})$. We can write any element of Kin the form $a + b\sqrt[3]{2} + c\sqrt[3]{4}$.

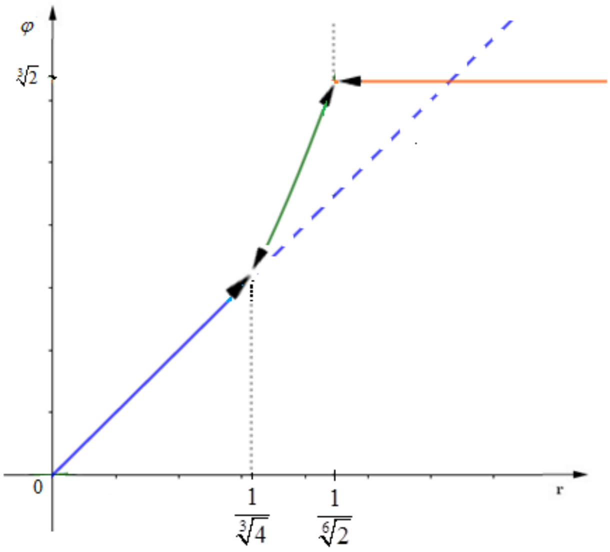
$$\mathbb{N}_{K\backslash\mathbb{Q}_2}(a + b\sqrt[3]{2} + c\sqrt[3]{4}) = a^3 + 4c^3 + 2b^3 - 6abc.$$

$$|\sqrt[3]{2}|_2 = \sqrt[3]{\mathbb{N}_{K\backslash\mathbb{Q}_2}(\sqrt[3]{2})} = \sqrt[3]{|2|_2} = \frac{1}{\sqrt[3]{2}}.$$

It follows that $|\breve{x}_1|_2 = |\breve{x}_2|_2 = \frac{1}{\sqrt[6]{2}}$, and for coefficient we get $\left|\frac{1}{\sqrt[3]{2}}\right| = \sqrt[3]{2}$. From this relation and equality (4) we can define the function $\varphi: 0,+\infty) \to 0,+\infty)$ by

$$\varphi(r) = \begin{cases} r, & \text{if } r < \frac{1}{\sqrt[3]{4}}, \\ \tilde{a}, & \text{if } r = \frac{1}{\sqrt[3]{4}}, \\ \sqrt[3]{4}r^2, & \text{if } \frac{1}{\sqrt[3]{4}} < r < \frac{1}{\sqrt[6]{2}}, \\ \tilde{b}, & \text{if } r = \frac{1}{\sqrt[6]{2}}, \\ \sqrt[3]{2}, & \text{if } r > \frac{1}{\sqrt[6]{2}}. \end{cases}$$

where $\tilde{a}$ and $\tilde{b}$ some positive numbers with $\tilde{a} < \frac{1}{\sqrt[3]{4}}$, and $\tilde{b} > \sqrt[3]{2}$. The graph of the function φ is



Picture 1.

Lemma 1. If $p = 2$ and $x \in S_r(x_1)$, then for the function (3) the following holds

$$|f^n(x)|_2 = \varphi^n(r)$$

By this lemma we see that the real dynamical system compiled from φ^n is directly related to the 2 −adic dynamical system $f^n(x), \ \ n \geq 1, \ \ x \in \mathbb{C}_2 \setminus P$.

The following lemma gives properties to this real dynamical system.

Lemma 2. The dynamical system generated by $\varphi(r)$ has the following properties:

1. $Fix(\varphi) = \left\{r: \ 0 \leq r < \frac{1}{\sqrt[3]{4}}\right\} \cup \left\{\frac{1}{\sqrt[3]{4}}: \ \text{if} \ \ \tilde{\alpha} = \frac{1}{\sqrt[3]{4}}\right\} \cup \{\sqrt[3]{2}\}$.

2. If $r > \frac{1}{\sqrt[3]{4}}$, then

$$\lim_{n\to\infty} \varphi^n(r) = \sqrt[3]{2}.$$

3. If $r = \frac{1}{\sqrt[3]{4}}$ and $\tilde{\alpha} < \frac{1}{\sqrt[3]{4}}$, then $\varphi^n(r) = \tilde{\alpha}$ for all $n \geq 1$.

Proof. 1. This is the result of a simple analysis of the equation $\varphi(r) = r$.

2. By definition of $\varphi(r)$, for $r > \frac{1}{\sqrt[6]{2}}$ we have $\varphi(r) = \sqrt[3]{2}$, i.e., the function is constant. For $r = \frac{1}{\sqrt[6]{2}}$ we have $\varphi\left(\frac{1}{\sqrt[6]{2}}\right) = \tilde{b} \geq \sqrt[3]{2}$ and thus we get $\varphi\left(\frac{1}{\sqrt[6]{2}}\right) > \frac{1}{\sqrt[6]{2}}$. Consequently,

$$\lim_{n\to\infty} \varphi^n\left(\frac{1}{\sqrt[6]{2}}\right) = \sqrt[3]{2}.$$

Assume now $\frac{1}{\sqrt[3]{4}} < r < \sqrt[3]{2}$ then $\varphi(r) = \sqrt[3]{4}r^2, \ \ \varphi'(r) = 2\sqrt[3]{4}r > 2$ and

$$\varphi\left(\left(\frac{1}{\sqrt[3]{4}}, \frac{1}{\sqrt[6]{2}}\right)\right) = \left(\frac{1}{\sqrt[3]{4}}, \sqrt[3]{2}\right) \cup \{\tilde{\alpha}\}.$$

Since $\varphi'(r) > 2$ for $r \in \left(\frac{1}{\sqrt[3]{4}}, \frac{1}{\sqrt[6]{2}}\right)$ there exists $n_0 \in \mathbb{N}$ such that $\varphi^{n_0}(r) \in \left(\frac{1}{\sqrt[6]{2}}, \sqrt[3]{2}\right)$. Hence for $n \geq n_0$ we get $\varphi^n(r) > \frac{1}{\sqrt[6]{2}}$ and consequently

$$\lim_{n\to\infty} \varphi^n(r) = \sqrt[3]{2}.$$

3. If $r = \frac{1}{\sqrt[3]{4}}$ and $\tilde{\alpha} < \frac{1}{\sqrt[3]{4}}$ then $\varphi(r) = \tilde{\alpha} < \frac{1}{\sqrt[3]{4}}$. Moreover, $\tilde{\alpha}$ is a fixed point for the function $\varphi(r)$. Thus for $n \geq 1$ we obtain $\varphi^n(r) = \tilde{\alpha}$.

By Lemma 1 and Lemma 2 we get

Theorem 2. The 2-adic dynamical system generated by function (3) has the following properties:

1. $SI(x_1) = U_{\frac{1}{\sqrt[3]{4}}}(0)$.

2. $x_2 \in S_{\sqrt[3]{2}}(0)$. The fixed point x_2 is attractive and

$$A(x_2) = \mathbb{C}_2 \setminus (V_{\frac{1}{\sqrt[3]{4}}}(0) \cup P).$$

3. If $x \in S_{\frac{1}{\sqrt[3]{4}}}(0)$, then there exists $\mu_1 < \frac{1}{\sqrt[3]{4}}$ such that $f^m(x) \in S_{\mu_1}(0)$ for any$m \geq 1$.

Proof. 1. By Lemma 1 and part 1 of Lemma 2 we see that spheres $S_r(0)$, $r < \frac{1}{\sqrt[3]{4}}$ and$S_{\sqrt[3]{2}}(0)$ are invariant for f. Thus$SI(x_1) = U_{\frac{1}{\sqrt[3]{4}}}(0)$. Consequently, $|x_2|_2 = \left|\frac{3}{\sqrt[3]{2}}\right|_2 = \sqrt[3]{2}$, i.e., $x_2 \in S_{\sqrt[3]{2}}(0)$.

2. In this case x_2 will be attractive fixed point, i.e.,

$$|f'(x_2)|_2 = |2\sqrt[3]{2}|_2 = \frac{1}{2\sqrt[3]{2}} < 1.$$

From Lemma 1 and part 2 of Lemma 2 we have

$$\lim_{n\to\infty} f^n(x) \in S_{\sqrt[3]{2}}(0)$$

for all $x \in S_r(0) \setminus P, \ \ r > \frac{1}{\sqrt[3]{4}}$.

Let $x \in S_{\sqrt[3]{2}}(0)$. We have

$$\left|f(x)-\frac{3}{\sqrt[3]{2}}\right|_2=\left|x-\frac{3}{\sqrt[3]{2}}\right|_2\cdot\frac{\left|-\sqrt[3]{4}x-\sqrt[3]{2}\right|_2}{\left|x^2-\sqrt[3]{4}x-\sqrt[3]{2}\right|_2}$$

By $\left|-\sqrt[3]{4}x-\sqrt[3]{2}\right|_2=\frac{1}{2\sqrt[3]{2}}$ and $|x-\breve{x}_2|_2=|x-\breve{x}_1|_2=\left|\frac{\sqrt{3}}{\sqrt[3]{2}}\right|_2=\sqrt[3]{2}$ we get $\left|f(x)-\frac{3}{\sqrt[3]{2}}\right|_2<\left|x-\frac{3}{\sqrt[3]{2}}\right|_2$ for any $x\in S_{\sqrt[3]{2}}(0)\backslash P$. Consequently,

$$\lim_{n\to\infty} f^n(x)=x_2,\text{ for all } x\in S_r(0)\backslash P,\quad r>\frac{1}{\sqrt[3]{4}},$$

i.e., $A(x_2)=\mathbb{C}_2\backslash(V_{\frac{1}{\sqrt[3]{4}}}(0)\cup P)$.

3. If $x\in S_{\frac{1}{\sqrt[3]{4}}}(0)$ then by (4) we have

$$|f(x)|_2=\frac{1}{\sqrt[3]{4}}\cdot\frac{\left|\frac{1}{\sqrt[3]{2}}x-\sqrt[3]{2}\right|_2}{\left(\frac{1}{\sqrt[6]{2}}\right)^2}<\frac{1}{\sqrt[3]{4}}.$$

Thus, there is $\mu_1<\frac{1}{\sqrt[3]{4}}$ such that $f^m(x)\in S_{\mu_1}(0)$ for any $m\geq 1$ (see part 1 of Lemma 2). Hence if $x\in S_{\frac{1}{\sqrt[3]{4}}}(0)$, then there exists $\mu_1<\frac{1}{\sqrt[3]{4}}$ such that $f^m(x)\in S_{\mu_1}(0)$ for any$m\geq 1$.

We note that

$$P=\bigcup_{k=o}^{\infty}P_k,\quad P_k=\left\{x\in\mathbb{C}_2: f^k(x)\in\{\breve{x}_1,\breve{x}_2\}\right\}.$$

Theorem 3. 1. $P_k\neq 0$, for any $k=0,1,2,\ldots$.

2. $P_k\subset S_{r_k}(0)$, where $r_k=\frac{1}{\sqrt[6]{2}}\cdot\left(\frac{1}{\sqrt[3]{2}}\right)^{\frac{2^k-1}{2^k}}$, $k=0,1,2,\ldots$.

Proof. 1. In case $k=0$ we have $P_0=\{\breve{x}_1,\breve{x}_2\}\neq\emptyset$.

Assume for $k=n$ that $P_n=\left\{x\in\mathbb{C}_p: f^n(x)\in\{\breve{x}_1,\breve{x}_2\}\right\}\neq\emptyset$.

Now for $k=n+1$ to prove $P_{n+1}=\left\{x\in\mathbb{C}_p: f^{n+1}(x)\in\{\breve{x}_1,\breve{x}_2\}\right\}\neq\emptyset$ we have to show that the following equation has at least one solution:

$$f^{n+1}(x)=\breve{x}_i,\text{ for some } i=1,2.$$

By our assumption $P_k\neq 0$, there exists $y\in P_n$ such that $f^n(y)\in\{\breve{x}_1,\breve{x}_2\}$. Now we show that there exists x such that $f(x)=y$. We note that the equation $f(x)=y$ can be written as

$$\left(\frac{1}{\sqrt[3]{2}}-y\right)x^2-(\sqrt[3]{2}-\sqrt[3]{4}y)x+\sqrt[3]{2}y=0.\quad(5)$$

Since $\breve{x}_1,\breve{x}_2\in S_{\frac{1}{\sqrt[6]{2}}}(0)$, by the Lemma 1 and the part1 of Lemma 2 we know that $S_{\sqrt[3]{2}}(0)$ is an invariant, consequently, $P\cap S_{\sqrt[3]{2}}(0)=\emptyset$. Thus $\frac{1}{\sqrt[3]{2}}\notin P$, hence, $\frac{1}{\sqrt[3]{2}}-y\neq 0$. Since $\mathbb{C}_2$ is algebraic closed the equation (5) has two solutions, say $x=t_1,t_2$. For $x\in\{t_1,t_2\}$ we get

$$f^{n+1}(x)=f^n(f(x))=f^n(y)\in\{\breve{x}_1,\breve{x}_2\}.$$

Hence $P_{n+1}\neq\emptyset$. Therefore, by induction we get $P_k\neq 0$, for any $k=0,1,2,\ldots$.

2. We know that $|\breve{x}_1|_2=|\breve{x}_2|_2=\frac{1}{\sqrt[6]{2}}$. By (4) and part 2 of Lemma 2 for $x\in S_{\frac{1}{\sqrt[6]{2}}}(0)$, $x\neq\breve{x}_{1,2}$ we have

$$\lim_{n\to\infty} f^n(x)\in S_{\sqrt[3]{2}}(0),$$

i.e., $S_{\frac{1}{\sqrt[6]{2}}}(0)\cap P=\{\breve{x}_1,\breve{x}_2\}=P_0$. Denoting $r_0=\frac{1}{\sqrt[6]{2}}$ we write $P_0\subset S_{r_0}(0)$.

For each $k=1,2,3,\ldots$ we want to find some r_k such that the solution x of $f^k(x)=\breve{x}_i$, (for some $i=1,2$.) belongs to $S_{r_k}(0)$, i.e., $x\in S_{r_k}(0)$. By Lemma 1 we should have

$$\psi^k_{\frac{1}{\sqrt[6]{2}}}(r_k)=\frac{1}{\sqrt[6]{2}}.$$

Now if we show that the last equation has unique solution r_k for each k, then we get

$$P_k=\left\{x\in\mathbb{C}_2: f^k(x)\in\{\breve{x}_1,\breve{x}_2\}\right\}\subset S_{r_k}(0).$$

By parts 1 and 3 of Lemma 2 we have $\frac{1}{\sqrt[3]{4}}<r_k\leq\frac{1}{\sqrt[6]{2}}$. Moreover, we have $r_0=\frac{1}{\sqrt[6]{2}}$ and $\frac{1}{\sqrt[3]{4}}<r_k<\frac{1}{\sqrt[6]{2}}$ for each $k=1,2,\ldots$. For such r_k, by definition of $\psi_{\frac{1}{\sqrt[6]{2}}}(r)$, we have

$$\psi_{\frac{1}{\sqrt[6]{2}}}(r_k)=\sqrt[3]{4}r_k^2.$$

Thus $\psi^k_{\frac{1}{\sqrt[6]{2}}}(r_k)=\frac{1}{\sqrt[6]{2}}$ has the form

$$\psi^k_{\frac{1}{\sqrt[6]{2}}}(r_k)=\frac{\sqrt[3]{2}^{2^k-1}}{\left(\frac{1}{\sqrt[6]{2}}\right)^{2(2^k-1)}}r_k^{2^k}=\frac{1}{\sqrt[6]{2}}$$

consequently,

$$r_k^{2^k}=\left(\frac{1}{\sqrt[6]{2}}\right)^{2^k}\cdot\left[\left(\frac{1}{\sqrt[3]{2}}\right)^{\frac{2^k-1}{2^k}}\right]^{2^k}.$$

Taking 2^k- root we obtain unique positive solution: $r_k=\frac{1}{\sqrt[6]{2}}\cdot\left(\frac{1}{\sqrt[3]{2}}\right)^{\frac{2^k-1}{2^k}}$.

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