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REACHING AN OPTIMAL SOLUTION TO A TRANSPORTATION PROBLEM INVOLVING A CONCAVE COST FUNCTION

Abstract: *The work is on reaching an optimal solution to a transportation problem involving a concave cost function with specific objectives to develop a new approach for solving optimization problems of a transportation problem in a concave case; demonstrate the effectiveness of the new approach using real life examples from published works, and comparing the self developed approach with the existing method. One Least Cost Row Column Difference Method (OLCRCDM) was employed to obtain the initial basic feasible solution. The transportation concave simplex technique was modified for a better solution and its steps were clearly stated in this study. Four numerical examples were employed to demonstrate the effectiveness of the developed technique in this study. The results revealed that out of the four numerical problems, the existing Karush-Kuhn-Tucker (KKT) procedure of Modified Distribution (MODI) method could not produce optimality point in the first example with North West Corner Method (NWCM) and Vogel Approximation Method (VAM) as a means of obtaining the IBFS, but the self developed did using OLCRCDM to obtain the IBFS and it yielded an optimal value of N253,000 with an optimal solution as* $z_{12}=13$, $z_{22}=5$, $z_{23}=8$, $z_{31}=11$, *and* $z_{33}=4$. *The remaining three examples were successfully solved with both the existing Karush-Kuhn-Tucker (KKT) procedure of MODI method and the new technique with optimal values of N377,000, GH¢ 236,000 and N509,000 respectively, but the new technique proved to be more efficient as it produced minimum number of iteration to optimality. The four problems were solved with Wolfram Mathematica and Anaconda Python programming softwares and the results agreed with the results obtained from the developed approach.*

Key words: *Concave Cost Function, OLCRCDM, Transportation Problem, Karush-Kuhn-Tucker, Optimal Solution, Proposed Algorithm.*

Language: *English*

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Introduction

Profit and cost is what virtually everyone deems interested in employing, using any efficient resources to optimize; hence various forms of transportation models are in existence. There are different kinds of transportation problems which are applied in the business world and the primary aim of a transportation problem is to find a means of moving this transfer of goods at a minimized total cost (Mostafa et al, 2022; Kaur & Kumar, 2011). In describing the transportation problem in its conventional form, the assumption is that an informed decision maker has an understanding on the value of transportation cost, demand and supply; hence, unpredictability is a common occurrence in real life circumstances.

This study is channeled to tackle a transportation problem in a concave cost function using a new approach. However, there are some factors that are responsible for the cost of goods, and some of them are transport, raw materials' costs and labour. This implies that the cost of raw materials is directly proportional to the cost of the goods, and the pricing system is also affected when there is a significant variation in the transportation cost (Rudi et al, 2016). The cost of goods per unit shipped is assumed to be constant irrespective of the quantity shipped from a given source to a defined destination; but the cost sometimes may not be constant in actuality. Sometimes, quantity discounts are feasible for large shipments in such a way that the marginal cost of transporting a unit might approach a specific pattern (Minken & Johansen, 2019).

Transportation problem involving a concave cost function, simply means a nonlinear transportation problem; indicating a scenario whereby volume discounts are being available for bulk shipments. In this case, the cost function of the transportation problem is separable, and the marginal cost (cost per unit of goods shipped) decreases as the shipment volume increases, so it generally assumes a concave structure. It will increase due to the increase in the total cost per additional unit of goods shipped (Haruna et al, 2012). Discounts may be directly related to the unit of commodity or may have the same rate for a particular amount. Thus, the discount may be either directly associated to the unit commodity or have the equivalent rate for some quantity. However, if the discount is directly associated to the unit commodity, then the resulting cost function becomes continuous and possesses continuous first partial derivatives.

Transportation Problem via Concave Cost Functions

Volume discounts may be available for bulk shipments. In this case, the cost function of the transportation problem is separable, and the marginal cost (cost per unit of goods shipped) decreases as the shipment volume increases, so we generally assume a concave structure. It will increase due to the increase in the total cost per additional unit of goods shipped (Haruna et al, 2012). However, If the discount is directly associated to the unit commodity, then the resulting cost function becomes continues and possesses continues first partial derivatives.

Given a function that is differentiable

K: $\mathbb{R}^{nm} \rightarrow \mathbb{R}$

The nonlinear transportation problem when the discount is directly related to the unit commodity is defined mathematically as stated in Equation (1)

$$\left.\begin{aligned} &Minimize\ K(Z) \\ &Subject\ to: AZ = b \\ &\qquad\qquad Z \geq 0 \end{aligned}\right\} \qquad (1)$$

where

$$Z = \begin{pmatrix} Z_{11} \\ Z_{12} \\ \vdots \\ Z_{mj} \\ \vdots \\ Z_{mn} \\ Z_{11} \\ Z_{21} \\ \vdots \\ Z_{in} \\ \vdots \\ Z_{mn} \end{pmatrix}; \quad b = \begin{pmatrix} S_1 \\ S_2 \\ \vdots \\ S_m \\ D_1 \\ D_2 \\ \vdots \\ D_n \end{pmatrix}; A = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Nonlinear Transportation Tableau

The nonlinear transportation tableau is defined as shown in Table 1.

Table 1: Nonlinear Transportation Tableau

	1	2	…	j	…	n		
1	$\frac{\partial K(\bar{Z})}{\partial Z_{11}}$	$\frac{\partial K(\bar{Z})}{\partial Z_{12}}$	…	$\frac{\partial K(\bar{Z})}{\partial Z_{1j}}$	…	$\frac{\partial K(\bar{Z})}{\partial Z_{1n}}$	S_1	R_1
2	$\frac{\partial K(\bar{Z})}{\partial Z_{21}}$	$\frac{\partial K(\bar{Z})}{\partial Z_{22}}$	…	$\frac{\partial K(\bar{Z})}{\partial Z_{2j}}$	…	$\frac{\partial K(\bar{Z})}{\partial Z_{2n}}$	S_2	R_2
⋮	…	…	…	…	…	⋮	⋮	⋮
i	$\frac{\partial K(\bar{Z})}{\partial Z_{i1}}$	$\frac{\partial K(\bar{Z})}{\partial Z_{i2}}$	…	$\frac{\partial K(\bar{Z})}{\partial Z_{ij}}$	…	$\frac{\partial K(\bar{Z})}{\partial Z_{in}}$	S_i	R_i
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
m	$\frac{\partial K(\bar{Z})}{\partial Z_{m1}}$	$\frac{\partial K(\bar{Z})}{\partial Z_{m2}}$	…	$\frac{\partial K(\bar{Z})}{\partial Z_{mj}}$	…	$\frac{\partial K(\bar{Z})}{\partial Z_{mn}}$	S_m	R_m
	D_1	D_2	…	D_j	…	D_n		
	T_1	T_2	…	T_j	…	T_n		

Where $\bar{Z}$ is the current basic solution

Solution Steps to Transportation Problem in the Concave Case (Self Developed)

Initialization

Obtain the initial basic solution via One Least Cost Row Column Difference Method (OLCRCDM).

Iteration

Step 1: Obtain $\frac{\partial f(\bar{z})}{\partial z_{B_{ij}}}$ for the occupied cells using the equation

$$\frac{\partial f(\bar{z})}{\partial z_{B_{ij}}} = \frac{\partial}{\partial z_{B_{ij}}}\left[k_{ij}z_{ij} - p_{ij}z_{ij}^2\right] at\ \bar{z} \qquad (2)$$

Step 2: Obtain $\frac{\partial f(\bar{z})}{\partial z_{ij}}$ for the unoccupied cells using the equation

$$\frac{\partial f(\bar{z})}{\partial z_{ij}} = \int_0^1 (k_{ij} - 2p_{ij}z_{ij})dz_{ij} \qquad (3)$$

$$= k_{ij}z_{ij} - p_{ij}z_{ij}^2\Big|_0^1$$

$$\therefore \frac{\partial f(\bar{z})}{\partial z_{ij}} = k_{ij} - p_{ij} \qquad (4)$$

Step 3: Obtain

Determine the values of r_i and t_j from the equation,

$$\frac{\partial w}{\partial z_{B_{ij}}} = \frac{\partial f(\bar{z})}{\partial z_{B_{ij}}} - t_j + r_i = 0 \Rightarrow \frac{\partial f(\bar{z})}{\partial z_{B_{ij}}} = t_j - r_i \quad (5)$$

Where z_{Bij} are the basic variables

Step 4: The reduced cost for the non-basic variables is obtained using the formula;

$$\frac{\partial w}{\partial z_{ij}} = \int_0^1 \frac{\partial f(\bar{z})}{\partial z_{ij}} dz_{ij} - \int_{r_i}^{t_j} dz_{ij} = \int_0^1 \frac{\partial f(\bar{z})}{\partial z_{ij}} dz_{ij} - t_j + r_i \qquad (6)$$

If $\int_0^1 \frac{\partial f(\bar{z})}{\partial z_{ij}} dz_{ij} - t_j + r_i \geq 0$

for all z_{ij} – non basic, stop, $\bar{z}$ is KKT point. Otherwise go to step 5.

Step 5: Compute

$$\frac{\partial w}{\partial z_{kl}} = \left\{\int_0^1 \frac{\partial f(\bar{z})}{\partial z_{ij}} dz_{ij} - t_j + r_i\right\} \qquad (7)$$

z_{kl} will enter the basic. Allocate $z_{kl} = \theta$ where θ is found as in the linear transportation case.

The constraints are satisfied when the allocations are adjusted.

The variable to be left say z_{Bkv}, is determined, while the variable which is basic that turns to zero first is z_{Bkv} while making the adjustment. Obtain the new variables for the basic, and move to step 1.

Numerical Problems

Example 1

Unilever Nigeria Plc located in Apapa Ikeja, produces and sells the products as indicated in Table 2:

Table 2: Market Segment Analysis

	MARKETS SEGMENTS			SUPPLY
	P	Q	R	
Omo washing powder	5	4	6	13,000
Blue Band margarine	7	6	5	13,000
Vaseline	9	11	8	15,000
DEMAND	11,000	18,000	12,000	

The company's percentage discount as a policy is presented in the Table 3:

Table 3: Company's Percentage Discount

	P	Q	R
Omo washing powder	0.03	0.015	0.04
Blue Band margarine	0.02	0.03	0.05
Vaseline	0.035	0.05	0.03

Source: Okenwe (2018)

The basic feasible solution, which was obtained using One Least Cost Row Column Difference Method (OLCRCDM) is Z_{12} = 13, Z_{22} = 5, Z_{23} = 8, Z_{31} = 11, Z_{33} = 4, which results in a transportation cost of = (0, 13, 0, 0, 5, 8, 11, 0, 4), in Thousands.

$$\bar{z} = (z_{11}, z_{B12}, z_{13}, z_{21}, z_{B22}, z_{B23}, z_{B31}, z_{32}, z_{B33})$$

The total transportation cost is

13000(4) + 5000(6) + 8000(5) + 11000(9) + 4000(8) = ₦ 253000

Due to the discount given to each box as a result of large volume of transporting from source i to destination j, the formulation of the transportation problem in nonlinear form is:

$$Min. \sum_{i=1}^{3}\sum_{j=1}^{3} k_{ij} z_{ij}$$

$$\begin{aligned} S.t. \quad & z_{11} + z_{12} + z_{13} = 13 \\ & z_{21} + z_{22} + z_{23} = 13 \\ & z_{31} + z_{32} + z_{33} = 15 \\ & z_{11} + z_{21} + z_{31} = 11 \\ & z_{12} + z_{22} + z_{32} = 18 \\ & z_{13} + z_{23} + z_{33} = 12 \end{aligned}$$

Where

$$k_{11}z_{11} = 5z_{11} - p_{11}z_{11}^2,\ k_{12}z_{12} = 4z_{12} - p_{12}z_{12}^2,\ k_{13}z_{13} = 6z_{13} - p_{13}z_{13}^2,\ k_{21}z_{21} = 7z_{21} - p_{21}z_{21}^2$$
$$k_{22}z_{22} = 6z_{22} - p_{22}z_{22}^2,\ k_{23}z_{23} = 5z_{23} - p_{23}z_{23}^2,\ k_{31}z_{31} = 9z_{31} - p_{31}z_{31}^2,\ k_{32}z_{32} = 11z_{32} - p_{32}z_{32}^2$$
$$k_{33}z_{33} = 8z_{33} - p_{33}z_{33}^2$$

Due to the discount given to each box as a result of large volume of transporting from source i to destination j, then the cost function (k_{ij}) is indicated as:

$$k_{11}z_{11} = 5z_{11} - 0.03z_{11}^2,\ k_{12}z_{12} = 4z_{12} - 0.015z_{12}^2,\ k_{13}z_{13} = 6z_{13} - 0.04z_{13}^2,\ k_{21}z_{21} = 7z_{21} - 0.02z_{21}^2$$
$$k_{22}z_{22} = 6z_{22} - 0.03z_{22}^2,\ k_{23}z_{23} = 5z_{23} - 0.05z_{23}^2,\ k_{31}z_{31} = 9z_{31} - 0.035z_{31}^2,\ k_{32}z_{32} = 11z_{32} - 0.05z_{32}^2$$
$$k_{33}z_{33} = 8z_{33} - 0.03z_{33}^2$$

The first derivations of the cost function k_{ij} are given as:

$$k_{11}z_{11} = 5 - 0.06z_{11},\ k_{12}z_{12} = 4 - 0.03z_{12},\ k_{13}z_{13} = 6 - 0.08z_{13},\ k_{21}z_{21} = 7 - 0.04z_{21},$$
$$k_{22}z_{22} = 6 - 0.06z_{22},\ k_{23}z_{23} = 5 - 0.1z_{23},\ k_{31}z_{31} = 9 - 0.07z_{31};\ k_{32}z_{32} = 11 - 0.1z_{32},\ k_{33}z_{33} = 8 - 0.06z_{33}$$

For the occupied cells, the first derivations at $\bar{z}$ are obtained using Equation (2) as:

$$\frac{\partial f(\bar{z})}{\partial z_{12}} = 3.61;\ \frac{\partial f(\bar{z})}{\partial z_{22}} = 5.7;\ \frac{\partial f(\bar{z})}{\partial z_{23}} = 4.2;\ \frac{\partial f(\bar{z})}{\partial z_{31}} = 8.23;\ \frac{\partial f(\bar{z})}{\partial z_{33}} = 7.76$$

For the unoccupied cells, the integration of the first derivations of the non-basic variables $\left[\int_0^1 \frac{\partial f(\bar{z})}{\partial z_{ij}} dz_{ij}\right]$ is obtained using Equation (3) as follows:

$$\frac{\partial f(\bar{z})}{\partial z_{11}} = \int_0^1 (5 - 0.06z_{11})dz_{11} = 5z_{11} - 0.03z_{11}^2\Big|_0^1 = 5 - 0.03 = 4.97$$

$$\frac{\partial f(\bar{z})}{\partial z_{13}} = \int_0^1 (6 - 0.08z_{13})dz_{13} = 6z_{13} - 0.04z_{13}^2\Big|_0^1 = 6 - 0.04 = 5.96$$

$$\frac{\partial f(\bar{z})}{\partial z_{21}} = 6.98;\ \frac{\partial f(\bar{z})}{\partial z_{32}} = 10.95$$

Now using Equation (5), we find

$$\frac{\partial w}{\partial z_{B_{ij}}} = \frac{\partial f(\bar{z})}{\partial z_{B_{ij}}} - t_j + r_i = 0 \Rightarrow \frac{\partial f(\bar{z})}{\partial z_{B_{ij}}} = t_j - r_i$$

Thus,

$$t_2 - r_1 = 3.61;\ t_2 - r_2 = 5.7;\ t_3 - r_2 = 4.2;\ t_1 - r_3 = 8.23;\ t_3 - r_3 = 7.76$$

$r_1 = 1$ (Number of basic cells in row one) and from the equations above; we have

$$r_1 = 2,\ r_2 = -1.09,\ r_3 = -4.65$$
$$t_1 = 3.58,\ t_2 = 4.61,\ t_3 = 3.11$$

The computation for the reduced costs of the non-basic variables is obtained using Equation (6) as follows;

$$\frac{\partial w}{\partial z_{ij}} = \int_0^1 \frac{\partial f(\bar{z})}{\partial z_{ij}} dz_{ij} - \int_{r_i}^{t_j} dz_{ij} = \int_0^1 \frac{\partial f(\bar{z})}{\partial z_{ij}} dz_{ij} - t_j + r_i.$$ That is

$$\frac{\partial w}{\partial z_{11}} = \int_0^1 \frac{\partial f(\bar{z})}{\partial z_{11}} dz_{11} - t_1 + r_1 = 2.39;\ \frac{\partial w}{\partial z_{13}} = \int_0^1 \frac{\partial f(\bar{z})}{\partial z_{13}} dz_{13} - t_3 + r_1 = 3.85;\ \frac{\partial w}{\partial z_{21}} = \int_0^1 \frac{\partial f(\bar{z})}{\partial z_{21}} dz_{21} - t_1 + r_2 = 2.31;$$

$$\frac{\partial w}{\partial z_{32}} = \int_0^1 \frac{\partial f(\bar{z})}{\partial z_{32}} dz_{32} - t_2 + r_3 = 1.69$$

Since all the non-basic variables of the reduced costs are positive, then the optimality point of $\bar{z}$ is reached, with a minimum cost of transportation as ₦253000.

Example 2

Bottling company in Imo State, Owerri plant, Nigeria, a distributor of different categories of drinks (Fanta, Coke, Sprite) in different market segments as indicated in Table 4.

Table 4: Categories of Drinks (Fanta, Coke, Sprite) in Different Market Segments

	MARKETS SEGMENTS					s_i
	Mbaise	Orlu	Aba	Umuahia	Afikpo	
Fanta	14	8	11	12	8	11,000
Coke	12	10	7	15	11	17,000
Sprite	10	9	14	13	15	11,000
d_j	6,000	7,000	9,000	10,000	7,000	

The policy of the company allows percentage discounts as shown in Table 5.

Table 5: Policy of the Company's Allowable Percentage Discounts

	Mbaise	Orlu	Aba	Umuahia	Afikpo
Fanta	0.01	0.04	0.02	0.04	0.02
Coke	0.03	0.01	0.02	0.013	0.03
Sprite	0.02	0.04	0.03	0.02	0.03

Source: Osuji et al (2014)

The basic feasible solution, which was obtained using One Least Cost Row Column Difference Method (OLCRCDM) is

$$Z_{14} = 4,\ Z_{15} = 7,\ Z_{22} = 7,\ Z_{23} = 9,\ Z_{24} = 1,\ Z_{31} = 6,\ Z_{34} = 5$$

which results in a transportation cost of

= (0, 0, 0, 4, 7, 0, 7, 9, 1, 0, 6, 0, 0, 5, 0), in Thousands.

$$\bar{z} = (z_{11}, z_{12}, z_{13}, z_{B14}, z_{B15}, z_{21}, z_{B22}, z_{B23}, z_{B24}, z_{25}, z_{B31}, z_{32}, z_{33}, z_{B34}, z_{35})$$

The total transportation cost is

4000(12) + 7000(8) + 7000(10) + 9000(7) + 1000(15) +6000(10) + 5000(13) = ₦ 377,000

Due to the discount given to each box as a result of large volume of transporting from source i to destination j, the formulation of the transportation problem in nonlinear form is:

$$Min. \sum_{i=1}^{3}\sum_{j=1}^{5} k_{ij} z_{ij}$$

$$\begin{aligned} S.t.\quad & z_{11} + z_{12} + z_{13} + z_{14} + z_{15} = 11 \\ & z_{21} + z_{22} + z_{23} + z_{24} + z_{25} = 17 \\ & z_{31} + z_{32} + z_{33} + z_{34} + z_{35} = 11 \\ & z_{11} + z_{21} + z_{31} = 6 \\ & z_{12} + z_{22} + z_{32} = 7 \\ & z_{13} + z_{23} + z_{33} = 9 \\ & z_{14} + z_{24} + z_{34} = 10 \\ & z_{15} + z_{25} + z_{35} = 7 \end{aligned}$$

where

$$k_{11}z_{11} = 14z_{11} - p_{11}z_{11}^2,\ k_{12}z_{12} = 8z_{12} - p_{12}z_{12}^2,\ k_{13}z_{13} = 11z_{13} - p_{13}z_{13}^2,\ k_{14}z_{14} = 12z_{14} - p_{14}z_{14}^2,$$
$$k_{15}z_{15} = 8z_{15} - p_{15}z_{15}^2, k_{21}z_{21} = 12z_{21} - p_{21}z_{21}^2,\ k_{22}z_{22} = 10z_{22} - p_{22}z_{22}^2,\ k_{23}z_{23} = 7z_{23} - p_{23}z_{23}^2,$$
$$k_{24}z_{24} = 15z_{24} - p_{24}z_{24}^2, k_{25}z_{25} = 11z_{25} - p_{25}z_{25}^2, k_{31}z_{31} = 10z_{31} - p_{31}z_{31}^2;\ k_{32}z_{32} = 9z_{32} - p_{32}z_{32}^2,$$
$$k_{33}z_{33} = 14z_{33} - p_{33}z_{33}^2,\ k_{34}z_{34} = 13z_{34} - p_{34}z_{34}^2, k_{35}z_{35} = 15z_{35} - p_{35}z_{35}^2$$

Due to the discount given to each box as a result of large volume of transporting from source i to destination j, then the cost function (k_{ij}) is indicated as:

$$k_{11}z_{11} = 14z_{11} - 0.01z_{11}^2,\ k_{12}z_{12} = 8z_{12} - 0.04z_{12}^2,\ k_{13}z_{13} = 11z_{13} - 0.02z_{13}^2,\ k_{14}z_{14} = 12z_{14} - 0.04z_{14}^2,$$
$$k_{15}z_{15} = 8z_{15} - 0.02z_{15}^2, k_{21}z_{21} = 12z_{21} - 0.03z_{21}^2,\ k_{22}z_{22} = 10z_{22} - 0.01z_{22}^2,\ k_{23}z_{23} = 7z_{23} - 0.02z_{23}^2,$$
$$k_{24}z_{24} = 15z_{24} - 0.013z_{24}^2, k_{25}z_{25} = 11z_{25} - 0.03z_{25}^2, k_{31}z_{31} = 10z_{31} - 0.02z_{31}^2;\ k_{32}z_{32} = 9z_{32} - 0.04z_{32}^2,$$
$$k_{33}z_{33} = 14z_{33} - 0.03z_{33}^2,\ k_{34}z_{34} = 13z_{34} - 0.02z_{34}^2, k_{35}z_{35} = 15z_{35} - 0.03z_{35}^2$$

The first derivations of the cost function k_{ij} are given as:

$$k_{11}z_{11} = 14 - 0.02z_{11},\ k_{12}z_{12} = 8 - 0.08z_{12},\ k_{13}z_{13} = 11 - 0.04z_{13},\ k_{14}z_{14} = 12 - 0.08z_{14}, k_{15}z_{15} = 8 - 0.04z_{15},$$
$$k_{21}z_{21} = 12 - 0.06z_{21},\ k_{22}z_{22} = 10 - 0.02z_{22},\ k_{23}z_{23} = 7 - 0.04z_{23},\ k_{24}z_{24} = 15 - 0.026z_{24}, k_{25}z_{25} = 10 - 0.06z_{25},$$
$$k_{31}z_{31} = 10 - 0.02z_{31};\ k_{32}z_{32} = 9 - 0.08z_{32},\ k_{33}z_{33} = 14 - 0.06z_{33},\ k_{34}z_{34} = 13 - 0.04z_{34}, k_{35}z_{35} = 15 - 0.06z_{35}$$

For the occupied cells, the first derivations at $\bar{z}$ are obtained using Equation (2) as:

$$\frac{\partial f(\bar{z})}{\partial z_{14}} = 11.68;\ \frac{\partial f(\bar{z})}{\partial z_{15}} = 7.72;\ \frac{\partial f(\bar{z})}{\partial z_{22}} = 9.86;\ \frac{\partial f(\bar{z})}{\partial z_{23}} = 6.64; \frac{\partial f(\bar{z})}{\partial z_{24}} = 14.97;\ \frac{\partial f(\bar{z})}{\partial z_{31}} = 9.76, \frac{\partial f(\bar{z})}{\partial z_{34}} = 12.8$$

For the unoccupied cells, the integration of the first derivations of the non-basic variables $\left[\int_0^1 \frac{\partial f(\bar{z})}{\partial z_{ij}} dz_{ij}\right]$ is obtained using Equation (3) as follows:

$$\frac{\partial f(\bar{z})}{\partial z_{11}} = \int_0^1 (14 - 0.02z_{11})dz_{11} = 14z_{11} - 0.01z_{11}^2\Big|_0^1 = 14 - 0.01 = 13.99$$

$$\frac{\partial f(\bar{z})}{\partial z_{12}} = \int_0^1 (8 - 0.08z_{12})dz_{12} = 8z_{12} - 0.04z_{12}^2\Big|_0^1 = 8 - 0.04 = 7.96$$

$$\frac{\partial f(\bar{z})}{\partial z_{13}} = 10.98; \frac{\partial f(\bar{z})}{\partial z_{21}} = 11.97; \frac{\partial f(\bar{z})}{\partial z_{25}} = 10.97; \frac{\partial f(\bar{z})}{\partial z_{32}} = 8.96, \frac{\partial f(\bar{z})}{\partial z_{33}} = 13.97; \frac{\partial f(\bar{z})}{\partial z_{35}} = 14.97$$

Now using Equation (5), we find

$$\frac{\partial w}{\partial z_{B_{ij}}} = \frac{\partial f(\bar{z})}{\partial z_{B_{ij}}} - t_j + r_i = 0 \Rightarrow \frac{\partial f(\bar{z})}{\partial z_{B_{ij}}} = t_j - r_i$$

Thus,

$$t_4 - r_1 = 11.68;\ t_5 - r_1 = 7.72;\ t_2 - r_2 = 9.86; t_3 - r_2 = 6.64; t_4 - r_2 = 14.97;\ t_1 - r_3 = 9.76;\ t_4 - r_3 = 12.8$$

$r_1 = 2$ (Number of basic cells in row one) and from the equations above; we have

$$r_1 = 2, r_2 = -1.29, r_3 = 0.88$$
$$t_1 = 10.64, t_2 = 8.57, t_3 = 5.35, t_4 = 13.68, t_5 = 9.72$$

The computation for the reduced costs of the non-basic variables is obtained via the formula in Equation (6) as;

$$\frac{\partial w}{\partial z_{ij}} = \int_0^1 \frac{\partial f(\bar{z})}{\partial z_{ij}} dz_{ij} - \int_{r_i}^{t_j} dz_{ij} = \int_0^1 \frac{\partial f(\bar{z})}{\partial z_{ij}} dz_{ij} - t_j + r_i. \text{ That is}$$

$$\frac{\partial w}{\partial z_{11}} = \int_0^1 \frac{\partial f(\bar{z})}{\partial z_{11}} dz_{11} - t_1 + r_1 = 5.35; \; \frac{\partial w}{\partial z_{12}} = \int_0^1 \frac{\partial f(\bar{z})}{\partial z_{12}} dz_{12} - t_2 + r_1 = 1.39; \; \frac{\partial w}{\partial z_{13}} = \int_0^1 \frac{\partial f(\bar{z})}{\partial z_{13}} dz_{13} - t_3 + r_1 = 7.63;$$

$$\frac{\partial w}{\partial z_{21}} = \int_0^1 \frac{\partial f(\bar{z})}{\partial z_{21}} dz_{21} - t_1 + r_2 = 0.04; \; \frac{\partial w}{\partial z_{25}} = \int_0^1 \frac{\partial f(\bar{z})}{\partial z_{25}} dz_{25} - t_5 + r_2 = -0.04; \; \frac{\partial w}{\partial z_{32}} = \int_0^1 \frac{\partial f(\bar{z})}{\partial z_{32}} dz_{32} - t_2 + r_3 = 1.27$$

$$\frac{\partial w}{\partial z_{33}} = \int_0^1 \frac{\partial f(\bar{z})}{\partial z_{33}} dz_{33} - t_3 + r_3 = 9.5; \; \frac{\partial w}{\partial z_{35}} = \int_0^1 \frac{\partial f(\bar{z})}{\partial z_{35}} dz_{35} - t_5 + r_3 = 6.13$$

$$Min.\left(\frac{\partial w}{\partial z_{ij}}\right) = \frac{\partial w}{\partial z_{25}} = -0.04$$

The existence of negative figure for the reduced cost implies non-optimality. Thus, the Table 6 is readjusted as indicated in step 5.

Hence, z_{25} enters the basis, and after adjustment of the values, z_{24} was removed from the basic.

Table 6: Adjustment of Table to Obtain the Leaving Variable

	Markets						
Sources	Mbaise	Orlu	Aba	Umuahia	Afikpo	s_i	r_i
Fanta	14 (5.35)	8 (1.39)	11 (7.63)	12 + 4	8 − 7	11000	2
Coke	12 (0.04)	10 7	7 9	15 1 −	11 (-0.04) +	17000	-1.29
Sprite	10 6	9 (1.27)	14 (9.5)	13 5	15 (6.13)	11000	0.88
d_j	6000	7000	9000	10000	7000		
t_j	10.64	8.57	5.35	13.68	9.72		

The leaving variable is the minimum figure among the corners with a minus sign for the basic variables in the loop. Therefore, the leaving variable is z_{24} since it has a minimum figure of 1. This implies that the corners with a plus sign would be increased by 1, while the corners with a minus sign would be reduced by 1. The Table is adjusted to produce the next Table as shown in Table 7:

Table 7: Summary of Table after Obtaining the Leaving Variable

	Markets					
Sources	Mbaise	Orlu	Aba	Umuahia	Afikpo	s_i
Fanta	14	8	11	12 (5)	8 (6)	11000
Coke	12	10 (7)	7 (9)	15	11 (1)	17000
Sprite	10 (6)	9	14	13 (5)	15	11000
d_j	6000	7000	9000	10000	7000	

The basic feasible solution at the conclusion of this level of iteration becomes;

$$\bar{z}^2 = (z_{11}, z_{12}, z_{13}, z_{B14}, z_{B15}, z_{21}, z_{B22}, z_{B23}, z_{24}, z_{B25}, z_{B31}, z_{32}, z_{33}, z_{B34}, z_{35})$$

The total transportation cost is

5000(12) + 6000(8) + 7000(10) + 9000(7) + 1000(11) +6000(10) + 5000(13) = ₦ 377,000

The first derivations of the cost function k_{ij} are given as:

$$k_{11}z_{11} = 14 - 0.02z_{11}, k_{12}z_{12} = 8 - 0.08z_{12}, k_{13}z_{13} = 11 - 0.04z_{13}, k_{14}z_{14} = 12 - 0.08z_{14}, k_{15}z_{15} = 8 - 0.04z_{15},$$
$$k_{21}z_{21} = 12 - 0.06z_{21}, k_{22}z_{22} = 10 - 0.02z_{22}, k_{23}z_{23} = 7 - 0.04z_{23}, k_{24}z_{24} = 15 - 0.026z_{24}, k_{25}z_{25} = 11 - 0.06z_{25},$$
$$k_{31}z_{31} = 10 - 0.02z_{31}; k_{32}z_{32} = 9 - 0.08z_{32}, k_{33}z_{33} = 14 - 0.06z_{33}, k_{34}z_{34} = 13 - 0.04z_{34}, k_{35}z_{35} = 15 - 0.06z_{35}$$

For the occupied cells, the first derivations at $\bar{z}$ are obtained using Equation (2):

$$\frac{\partial f(\bar{z})}{\partial z_{14}} = 11.6; \frac{\partial f(\bar{z})}{\partial z_{15}} = 7.76; \frac{\partial f(\bar{z})}{\partial z_{22}} = 9.86; \frac{\partial f(\bar{z})}{\partial z_{23}} = 6.64; \frac{\partial f(\bar{z})}{\partial z_{25}} = 10.94; \frac{\partial f(\bar{z})}{\partial z_{31}} = 9.76; \frac{\partial f(\bar{z})}{\partial z_{34}} = 12.8$$

For the unoccupied cells, the integration of the first derivations of the non-basic variables $\left[\int_0^1 \frac{\partial f(\bar{z})}{\partial z_{ij}} dz_{ij}\right]$ is obtained using Equation (4) as follows:

$$\frac{\partial f(\bar{z})}{\partial z_{11}} = \int_0^1 (14 - 0.02z_{11})dz_{11} = 14z_{11} - 0.01z_{11}^2\Big|_0^1 = 14 - 0.01 = 13.99$$

$$\frac{\partial f(\bar{z})}{\partial z_{12}} = \int_0^1 (8 - 0.08z_{12})dz_{12} = 8z_{12} - 0.04z_{12}^2\Big|_0^1 = 8 - 0.04 = 7.96$$

$$\frac{\partial f(\bar{z})}{\partial z_{13}} = 10.98; \frac{\partial f(\bar{z})}{\partial z_{21}} = 11.97; \frac{\partial f(\bar{z})}{\partial z_{24}} = 14.99; \frac{\partial f(\bar{z})}{\partial z_{32}} = 8.96; \frac{\partial f(\bar{z})}{\partial z_{33}} = 13.97; \frac{\partial f(\bar{z})}{\partial z_{35}} = 14.97$$

Now using Equation (5), we find

$$\frac{\partial w}{\partial z_{B_{ij}}} = \frac{\partial f(\bar{z})}{\partial z_{B_{ij}}} - t_j + r_i = 0 \Rightarrow \frac{\partial f(\bar{z})}{\partial z_{B_{ij}}} = t_j - r_i$$

Thus,

$$t_4 - r_1 = 11.6;\ t_5 - r_1 = 7.76;\ t_2 - r_2 = 9.86; t_3 - r_2 = 6.64; t_5 - r_2 = 10.94;\ t_1 - r_3 = 9.76;$$

$$t_4 - r_3 = 12.8$$

$r_1 = 2$ (Number of basic cells in row one) and from the equations above; we have

$$r_1 = 2,\ r_2 = -1.18,\ r_3 = 0.8$$

$$t_1 = 10.56,\ t_2 = 8.68,\ t_3 = 5.46,\ t_4 = 13.6,\ t_5 = 9.76$$

The computation for the reduced costs of the non-basic variables is obtained via the formula in Equation (6) as;

$$\frac{\partial w}{\partial z_{ij}} = \int_0^1 \frac{\partial f(\bar{z})}{\partial z_{ij}} dz_{ij} - \int_{r_i}^{t_j} dz_{ij} = \int_0^1 \frac{\partial f(\bar{z})}{\partial z_{ij}} dz_{ij} - t_j + r_i. \text{ That is}$$

$$\frac{\partial w}{\partial z_{11}} = \int_0^1 \frac{\partial f(\bar{z})}{\partial z_{11}} dz_{11} - t_1 + r_1 = 5.43;\ \frac{\partial w}{\partial z_{12}} = \int_0^1 \frac{\partial f(\bar{z})}{\partial z_{12}} dz_{12} - t_2 + r_1 = 1.28;\ \frac{\partial w}{\partial z_{13}} = \int_0^1 \frac{\partial f(\bar{z})}{\partial z_{13}} dz_{13} - t_3 + r_1 = 7.52;$$

$$\frac{\partial w}{\partial z_{21}} = \int_0^1 \frac{\partial f(\bar{z})}{\partial z_{21}} dz_{21} - t_1 + r_2 = 0.23;\ \frac{\partial w}{\partial z_{24}} = \int_0^1 \frac{\partial f(\bar{z})}{\partial z_{24}} dz_{24} - t_4 + r_2 = 0.21;\ \frac{\partial w}{\partial z_{32}} = \int_0^1 \frac{\partial f(\bar{z})}{\partial z_{32}} dz_{32} - t_2 + r_3 = 1.08;$$

$$\frac{\partial w}{\partial z_{33}} = \int_0^1 \frac{\partial f(\bar{z})}{\partial z_{33}} dz_{33} - t_3 + r_3 = 9.31;\ \frac{\partial w}{\partial z_{35}} = \int_0^1 \frac{\partial f(\bar{z})}{\partial z_{35}} dz_{35} - t_5 + r_3 = 6.01$$

Since all the non-basic variables of the reduced costs are positive, then the optimality point of $\bar{z}^2$ is reached.

Total cost for transportation = 5000(12) + 6000(8) + 7000(10) + 9000(7) + 1000(11) +6000(10) + 5000(13) = ₦ 377,000

Example 3 extracted from Abdul-Salam (2014) and Example 4 extracted from Opara et al. (2015) are solved via the same method as illustrated in Examples one and two, and their results are presented in Table 8, along with the results of others.

Table 8: Summary of Results for the Four Practical Examples Used

		NWCM	**LCM**	**VAM**	**Proposed**
Example 1	IBFS	270,000	261,000	270,000	253,000
	No. of Iteration to Optimality	Not determined	2	Not determined	1
	Optimal Solution	Did not produce Optimal	z_{12}=13, z_{22}=5, z_{23} =8, z_{31}=11, z_{33}=4	Did not produce Optimal	z_{12}=13, z_{22}=5, z_{23} =8, z_{31}=11, z_{33}=4
	Optimal Value	Not determined	253,000	Not determined	253,000
	Wolfram Mathematica Optimal Solution	$z_{12} = 13,\ z_{22} = 5,\ z_{23} = 8,\ z_{31} = 11,\ z_{33} = 4$			
	Anaconda Python Optimal Solution	$z_{12} = 13,\ z_{22} = 5,\ z_{23} = 8,\ z_{31} = 11,\ z_{33} = 4$			
	Optimal Value	**253,000**			

Example 2	IBFS	454,000	384,000	381,000	377,000
	No. of Iteration to Optimality	5	4	2	2
	Optimal Solution	z_{14}=4, z_{15}=7, z_{21}=1, z_{22}=7, z_{23}=9, z_{31}=5, z_{34}=6	z_{14}=5, z_{15}=6, z_{22}=7, z_{23}=9, z_{25}=1, z_{31}=6, z_{34}=5	z_{14}=4, z_{15}=7, z_{21}=1, z_{22}=7, z_{23}=9, z_{31}=5, z_{34}=6	z_{14}=5, z_{15}=6, z_{22}=7, z_{23}=9, z_{25}=1, z_{31}=6, z_{34}=5
	Optimal Value	377,000	377,000	377,000	377,000
	Wolfram Mathematica Optimal Solution	$z_{14}=5, z_{15}=6, z_{22}=7, z_{23}=9, z_{25}=1, z_{31}=6, z_{34}=5$			
	Anaconda Python Optimal Solution	$z_{14}=5, z_{15}=6, z_{22}=7, z_{23}=9, z_{25}=1, z_{31}=6, z_{34}=5$			
	Optimal Value	**377,000**			
Example 3	IBFS	420,000	264,000	236,000	236,000
	No. of Iteration to Optimality	5	2	1	1
	Optimal Solution	z_{12}=7, z_{13}=8, z_{21}=10, z_{22}=3, z_{24}=12, z_{31}=10	z_{12}=7, z_{13}=8, z_{21}=10, z_{22}=3, z_{24}=12, z_{31}=10	z_{12}=7, z_{13}=8, z_{21}=10, z_{22}=3, z_{24}=12, z_{31}=10	z_{12}=7, z_{13}=8, z_{21}=10, z_{22}=3, z_{24}=12, z_{31}=10
	Optimal Value	236,000	236,000	236,000	236,000
	Wolfram Mathematica Optimal Solution	$z_{12}=7, z_{13}=8, z_{21}=10, z_{22}=3, z_{24}=12, z_{31}=10$			
	Anaconda Python Optimal Solution	$z_{12}=7, z_{13}=8, z_{21}=10, z_{22}=3, z_{24}=12, z_{31}=10$			
	Optimal Value	**236,000**			
Example 4.4	IBFS	605,000	517,000	513,000	517,000
	No. of Iteration to Optimality	6	2	2	2
	Optimal Solution	z_{12}=8, z_{15}=5, z_{23}=12, z_{25}=4, z_{31}=9, z_{32}=2, z_{41}=1, z_{44}=14	z_{12}=8, z_{15}=5, z_{23}=12, z_{25}=4, z_{31}=9, z_{32}=2, z_{41}=1, z_{44}=14	z_{12}=4, z_{15}=9, z_{21}=4, z_{23}=12, z_{31}=5, z_{32}=6, z_{41}=1, z_{44}=14	z_{12}=8, z_{15}=5, z_{23}=12, z_{25}=4, z_{31}=9, z_{32}=2, z_{41}=1, z_{44}=14
	Optimal Value	509,000	509,000	509,000	509,000
	Wolfram Mathematica Optimal Solution	$z_{12}=8, z_{15}=5, z_{23}=12, z_{25}=4, z_{31}=9, z_{32}=2, z_{41}=1, z_{44}=14$			
	Anaconda Python Optimal Solution	$z_{12}=8, z_{15}=5, z_{23}=12, z_{25}=4, z_{31}=9, z_{32}=2, z_{41}=1, z_{44}=14$			
	Optimal Value	**509,000**			

Conclusion

The study developed an algorithm that can be used to solve transportation problem in a concave function. Four real life examples extracted from different authors of published works were employed successfully to demonstrate the effectiveness of the new approach as the results obtained are in agreement with the results obtained from using programming software packages. Comparison of the results obtained from the new approach with that of the existing algorithm showed that the new approach is more effective than the existing algorithm as it was able to solve a problem that the existing algorithm could not solve as was revealed in example one of this study, and also it is faster to optimal solution based on the number of iterations to optimality.

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