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Issue Article

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FINDING PSEUDOSOLUTIONS OF A LINEAR SYSTEM OF ALGEBRAIC EQUATIONS IN MAPLE

***Abstract**: An inconsistent m of linear algebraic n equations have no solution in the usual sense. In this connection, the notion of a pseudo-solution of the system is introduced, and together with it, the methods of finding such solutions are considered. To date, an urgent scientific direction - computer mathematics - has received a rapid development of systems. Defining it as a set of, algorithmic, hardware and software tools designed for efficient solving on a computer of all kinds of mathematical problems, including symbolic transformation and calculations with a high degree of visualization of all kinds of calculations considered finding pseudo-solutions in these systems. The use of computer mathematics systems will significantly expand the possibilities of automating all stages of mathematical modeling.*

***Key words**: pseudo-matrix, pseudo-solution, normal pseudo-solution, least squares method, skeleton decomposition of matrix, rank of matrix.*
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Introduction

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The solution of many problems of mathematical physics and mathematical modeling of complex physical and chemical phenomena and processes are reduced to the solution of appropriate systems of linear algebraic equations. In this case, there is a need to take into account the degree of influence of inaccuracies in the initial data on the result of the obtained solutions and in the case where systems of equations with imprecise initial data have an exact absolute solution. Because of this, the influence of inaccuracies in the initial data has a specific nature and even minor inaccuracies of random nature can lead to the fact that the obtained solutions and the true ones will have a strong difference. In case the system is inconsistent, the mathematical model is corrected and a pseudo-solution or a generalized solution of the system is considered. One of the methods of finding a pseudo-solution of the system is the method of least squares.

Let us consider a system of linear algebraic equations that is inconsistent:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m \end{cases} \quad (1)$$

where $x_1, x_2, ..., x_n$ - unknown, a_{ij} - coefficients at the unknowns, $b_1, b_2, ..., b_m$ - free terms, which in matrix form is:

$$A \cdot X = B, \quad (2)$$

where $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$ - system matrix, $B = \begin{pmatrix} b_1 \\ b_2 \\ \cdots \\ b_m \end{pmatrix}$ - matrix of free terms, $X = \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix}$ - matrix of unknowns.

He problem of finding such values of the unknowns $(x_1, x_2, ..., x_n)$ that the function $F(x)$ form [1]:

$$F(x) = |Ax - b|^2 = \sum_{i=1}^{m} \left| \sum_{j=1}^{n} a_{ij} x_j \right|^2, \qquad (3)$$

where $(x_1, x_2, ..., x_n)$ takes the smallest value and is the essence of the method of least squares, and $(x_1, x_2, ..., x_n)$ in this case represents pseudosolutions of the considered system (1)-(2). To find pseudosolutions, we consider the system [1]:

$$A^T \cdot A \cdot X = A^T \cdot B, \qquad (4)$$

which is obtained from the original system (1)-(2) by multiplication of the transpose matrix A^T to the original matrix A. We find a general solution of system (4), which is a general pseudosolution of system (1)-(2).System (4) is called a system of normal equations and it is always joint. A pseudosolution that has minimal norm is defined as a normal pseudosolution of the system (1)-(2) [2].

Another method of finding pseudo-solutions of the SLAE is based on the application of the pseudo-matrix to the original matrix of the system. In this case, the general pseudo-solution of the system (1)-(2) can be represented in the form [2]:

$$X_{обнеод} = A^+ \cdot B + (E - A^+ \cdot A) \cdot C, \qquad (5)$$

where $A^+ \cdot B = X^0$ is a normal pseudosolution of the system, $(E - A^+ \cdot A) \cdot C = X_{обнод}$ is a general pseudosolution of the homogeneous system corresponding to system (1)-(2), B is a matrix of free terms, E is a unit matrix corresponding to the order, C is a vector of arbitrary constants [2].

The pseudo-matrix exists for any matrix and is the only one.

If the rank of matrix A coincides with the number of its rows, i.e. $r(A) = m$, then the pseudo matrix A^+ has the form:

$$A^+ = A^T \cdot (A \cdot A^T)^{-1}, \qquad (6)$$

where A^T is a transposed matrix. If the rank of matrix A is the same as the number of its columns, i.e. $r(A) = n$, then the pseudomatrix A^T has the form:

$$A^+ = (A^T \cdot A)^{-1} \cdot A^T, \qquad (7)$$

In the general case, if the matrix A is represented by a skeleton decomposition: $A = B \cdot C$, then A^+ has the form:

$$A^+ = C^T (C \cdot C^T)^{-1} \cdot (B^T \cdot B)^{-1} \cdot B^T. \qquad (8)$$

To solve the problems of linear algebra in modern systems of computer mathematics there are libraries of programs, the implementation of the algorithms of which is based on mathematical reasoning. The possibility of solving such problems in the Maple system arises, taking into account the labor intensity of matrix calculations, which increases with increasing dimensionality of matrices. In this .article, we will implement the method of least squares for finding a pseudo-solution of systems of linear algebraic equations in the Maple system. We will investigate and find the solution of the system of linear algebraic equations:

$$\begin{cases} x_1 - x_2 = 6, \\ -x_1 + 2x_2 + x_3 = 6, \\ 2x_1 - 3x_2 - x_3 = -9, \\ x_2 + x_3 = 18. \end{cases}$$

We set the initial data and connect the specialized package for solving linear algebra problems in Maple: *LinearAlgebra* [3]:

```
restart;
with(LinearAlgebra);
a11 := 1 : a12 := -1 : a13 := 0 : b1 := 6 : a21 := -1 : a22 := 2 : a23 := 1 : b2 := 9 :
a31 := 2 : a32 := -3 : a33 := -1 : b3 := -9 : a41 := 0 : a42 := 1 : a43 := 1 : b4 := 18 :
s1 := {a11·x1 + a12·x2 + a13·x3 = b1, a21·x1 + a22·x2 + a23·x3 = b2, a31·x1 + a32·x2
      + a33·x3 = b3, a41·x1 + a42·x2 + a43·x3 = b4};
A := Matrix(4, 3, [a11, a12, a13, a21, a22, a23, a31, a32, a33, a41, a42, a43]); rA
      := Rank(A);
B := Matrix(4, 4, [a11, a12, a13, b1, a21, a22, a23, b2, a31, a32, a33, b3, a41, a42, a43, b4]);
    rB := Rank(B);
n := ColumnDimension(A);
```

To investigate this system we will use the Kronecker-Capelli theorem, for this purpose the program will contain 3 cycles. The first cycle establishes the coherence of the original system [3]:

$$\textbf{if } rB = rA \textbf{ then } rs := rB;\ print\,('SLAY\_sovmestnaya');\ \textbf{else}\ print('SLAY\_nesovmestnaya');\ \textbf{fi};$$

The second circular path determines the solution of a certain SLAE, which is possible if the ranks of the matrix of the system A and the extended matrix of the system B and the number of unknowns are equal [3]:

$$\begin{aligned}
&\textbf{if } rB = rA \textbf{ and } rB = n \textbf{ then } print('Sistema_sovmestna_i_opredelennaya');\\
&B1 := Matrix(4, 1, [\mathrm{b1}, \mathrm{b2}, \mathrm{b3}, \mathrm{b4}]);\\
&ObA := MatrixInverse(A);\\
&X := MatrixMatrixMultiply(ObA, B1);\\
&x11 := X[1, 1]; x22 := X[2, 1]; x33 := X[3, 1]; x44 := X[4, 1];\\
&PR := subs(x1 = x11, x2 = x22, x3 = x33, x4 = x44, s);\ \textbf{fi};
\end{aligned}$$

The third cycle is responsible for finding a solution to the uncertain system, which is found if the ranks of matrix of the system A and the extended matrix of the system B are equal, but these ranks are less than the number of unknowns [3]:

$$\begin{aligned}
&\textbf{if } rB = rA \textbf{ and } rB < n \textbf{ then } print('Sistema_sovmestna_i_ne_opredelennaya');\\
&B1 := Matrix(4, 1, [\mathrm{b1}, \mathrm{b2}, \mathrm{b3}, \mathrm{b4}]);\\
&X := LinearSolve(A, B1);\\
&x11 := X[1, 1]; x22 := X[2, 1]; x33 := X[3, 1]; x44 := X[4, 1];\\
&PR := subs(x1 = x11, x2 = x22, x3 = x33, x4 = x44, s1);\ \textbf{fi};
\end{aligned}$$

As a result of the program we have:

$$s1 := \{x1 - x2 = 6, x2 + x3 = 18, -x1 + 2x2 + x3 = 9, 2x1 - 3x2 - x3 = -9\}$$

$$A := \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 2 & -3 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$rA := 2$$

$$B := \begin{bmatrix} 1 & -1 & 0 & 6 \\ -1 & 2 & 1 & 9 \\ 2 & -3 & -1 & -9 \\ 0 & 1 & 1 & 18 \end{bmatrix}$$

$$rB := 3$$

$$n := 3$$

$$SLAY\_nesovmestnaya$$

As we see, the original SLAE is inconsistent. To implement the method of least squares it is necessary to perform matrix calculations, according to equality (4): 1) transpose the matrix of the system: At; 2) multiply matrices At and AX on the left, preliminarily entering matrix X; 3) multiply At and $B1$ on the left, where $B1$ is the matrix of free members; 4) make a system of linear algebraic equations, the elements of the left part of which will be the result of multiplication of matrices of the left part of equality (4): $A^T \cdot A \cdot X$, the elements of the right part are the result of multiplication of the matrices of the right part of equality (4) [1]: $A^T \cdot B$. These calculations are performed in the first cycle of the program described above [3]:

```
if rB = rA then rs := rB; print ('SLAY_sovmestnaya');
else print('SLAY_nesovmestnaya');
At := Transpose(A);
X := Matrix(3, 1, [x1,x2,x3]);
AX := MatrixMatrixMultiply(A,X);
AtAX := MatrixMatrixMultiply(At,AX);
B1 := Matrix(4, 1, [b1, b2, b3, b4]);
AtB1 := MatrixMatrixMultiply(At,B1);
s2 := {AtAX[1, 1]=AtB1[1, 1], AtAX[2, 1]=AtB1[2, 1], AtAX[3, 1]=AtB1[3, 1]};
fi;
```

As a result of the program functioning, we have:

$$AtAX := \begin{bmatrix} 6\,x1 - 9\,x2 - 3\,x3 \\ -9\,x1 + 15\,x2 + 6\,x3 \\ -3\,x1 + 6\,x2 + 3\,x3 \end{bmatrix}$$

$$AtB1 := \begin{bmatrix} -21 \\ 57 \\ 36 \end{bmatrix}$$

$$s2 := \{-9\,x1 + 15\,x2 + 6\,x3 = 57,\ -3\,x1 + 6\,x2 + 3\,x3 = 36,\ 6\,x1 - 9\,x2 - 3\,x3 = -21\}$$

We got the system $s2$:

$$\begin{cases} -9x_1 + 15x_2 + 6x_3 = 57, \\ -3x_1 + 6x_2 + 3x_3 = 36, \\ 6x_1 - 9x_2 - 3x_3 = -21. \end{cases}$$

For the system $s2$ we use the program of research and finding solutions of the SLAE according to the Kronecker - Capelli theorem, according to which the set of pseudosolutions of the original system $x11$, $x22$, $x33$ is obtained:

$$s1 := \{-9\,x1 + 15\,x2 + 6\,x3 = 57,\ -3\,x1 + 6\,x2 + 3\,x3 = 36,\ 6\,x1 - 9\,x2 - 3\,x3 = -21\}$$

$$A := \begin{bmatrix} -9 & 15 & 6 \\ -3 & 6 & 3 \\ 6 & -9 & -3 \end{bmatrix}$$

$$rA := 2$$

$$B := \begin{bmatrix} -9 & 15 & 6 & 57 \\ -3 & 6 & 3 & 36 \\ 6 & -9 & -3 & -21 \end{bmatrix}$$

$$rB := 2$$

$$n := 3$$

$$rs := 2$$

print SLAY_sovmestnaya

Sistema_sovmestna_i_ne_opredelennaya

$$B1 := \begin{bmatrix} 57 \\ 36 \\ -21 \end{bmatrix}$$

$$X := \begin{bmatrix} 22 - \_t_{1,1} \\ 17 - \_t_{1,1} \\ \_t_{1,1} \end{bmatrix}$$

$$x11 := 22 - \_t_{1,1}$$

$$x22 := 17 - \_t_{1,1}$$

$$x33 := \_t_{1,1}$$

To obtain a normal pseudo-solution in the third cycle: 1) we form a function $F(x)$, using (3); 2) we find the minimum point of the function $F(x)$, with respect to the free unknown ; 3) we substitute the minimum point in the found solutions [1], [4]:

```
if rB = rA and rB < n then print('Sistema_sovmestnaya_i_ne_opredelennaya');
B1 := Matrix(3, 1, [b1, b2, b3]);
X := LinearSolve(A, B1);
x11 := X[1, 1]; x22 := X[2, 1]; x33 := X[3, 1];
F := (x11)^2 + (x22)^2 + (x33)^2;
Fc := subs(_t[1, 1] = c, F);
Diff(Fc, c) = diff(Fc, c);
c1 := solve(diff(Fc, c) = 0, c);
x111 := subs(_t[1, 1] = c1, x11);
x222 := subs(_t[1, 1] = c1, x22);
x333 := subs(_t[1, 1] = c1, x33);
PR := subs(x1 = x111, x2 = x222, x3 = x333, s1);
fi;
```

$$F := (22 - \_t_{1,1})^2 + (17 - \_t_{1,1})^2 + \_t_{1,1}^2$$

$$Fc := (22 - c)^2 + (17 - c)^2 + c^2$$

$$\frac{d}{dc}\left((22 - c)^2 + (17 - c)^2 + c^2\right) = -78 + 6c$$

$$c1 := 13$$

$$x111 := 9$$

$$x222 := 4$$

$$x333 := 13$$

$$PR := \{-21 = -21, 36 = 36, 57 = 57\}$$

A normal pseudo-solution of the original system is obtained:

$$X = \begin{pmatrix} 9 \\ 14 \\ 13 \end{pmatrix}.$$

Let us consider finding pseudo-solutions of the SLAE using the pseudo-matrix system in the Maple system of computer mathematics. Enter the initial data and connect the specialized package for solving linear algebra problems in Maple: LinearAlgebra:

```
restart;
with(LinearAlgebra);
a11 := 1 : a12 := -1 : a13 := 0 : b1 := 6 : a21 := -1 : a22 := 2 : a23 := 1 : b2 := 9 :
a31 := 2 : a32 := -3 : a33 := -1 : b3 := -9 : a41 := 0 : a42 := 1 : a43 := 1 : b4 := 18 :
s1 := {a11·x1 + a12·x2 + a13·x3 = b1, a21·x1 + a22·x2 + a23·x3 = b2, a31·x1 + a32·x2
    + a33·x3 = b3, a41·x1 + a42·x2 + a43·x3 = b4};
A := Matrix(4, 3, [a11, a12, a13, a21, a22, a23, a31, a32, a33, a41, a42, a43]);
B1 := Matrix(4, 1, [b1, b2, b3, b4]);
```

Considering the skeleton decomposition of the system matrix, we calculate the rank, determine the number of rows and columns of the matrix of A system [3]:

```
rA := Rank(A); mA := RowDimension(A); nA := ColumnDimension(A);
```

Let us make the first cycle in which the rank of matrix A is equal to the number of its rows. According to the theory, we assume $C = A$, then 1) multiply matrix C and transpose matrix C^T and get matrix CC^T, 2) find inverse matrix OCC^T for matrix CC^T, 3) make matrix PC by multiplying matrix C^T and OCC^T, 3) make matrix B by multiplying matrix A and PC, thus skeleton expansion of matrix $A = B \cdot C$ is obtained, where B is unit matrix, and $C = A$. Let us compose the pseudo matrix PsA by the algorithm of formula (6) , [5],[6]:

$$
\begin{aligned}
&\textbf{if } rA = mA \textbf{ then } C := A;\\
&CCT := Multiply(C, Transpose(C));\\
&OCCT := MatrixInverse(CCT);\\
&PC := Multiply(Transpose(C), OCCT);\\
&B := Multiply(A, PC);\\
&A1 := Multiply(B, C);\\
&BTB := Multiply(Transpose(B), B);\\
&OBTB := MatrixInverse(BTB);\\
&PB := Multiply(OBTB, Transpose(B));\\
&PsA := Multiply(PC, PB);\\
&\textbf{fi};
\end{aligned}
$$

By analogy let us make a second cycle in which the rank of matrix A is equal to the number of columns. The pseudo matrix PsA will be made by the algorithm of formula (7) [1]:

$$
\begin{aligned}
&\textbf{if } rA = nA \textbf{ then } C := A;\\
&CTC := Multiply(Transpose(C), C);\\
&OCTC := MatrixInverse(CTC);\\
&PC := Multiply(OCTC, Transpose(C));\\
&B := Multiply(PC, A);\\
&A1 := Multiply(C, B);\\
&BTB := Multiply(Transpose(B), B);\\
&OBTB := MatrixInverse(BTB);\\
&PB := Multiply(OBTB, Transpose(B));\\
&PsA := Multiply(PC, PB);\\
&\textbf{fi};
\end{aligned}
$$

In the general case, let us form the third cycle, in which $r(A) \neq m$ and $r(A) \neq n$ and $n-m=1$, unlike the first and second cycles, the matrix G, which is reduced to the triangular form by the Gauss elimination method, and the matrix C will be obtained by deleting the row equal to m (the number of rows of matrix A). The pseudo matrix PsA is formed by the algorithm of formula (8) [1]:

$$
\begin{aligned}
&\textbf{if } rA \neq nA \textbf{ and } rA \neq mA \textbf{ and } nA - mA = 1 \textbf{ then } G := GaussianElimination(A);\\
&C := DeleteRow(G, mA);\\
&rC := Rank(C);\\
&CCT := Multiply(C, Transpose(C));\\
&OCCT := MatrixInverse(CCT);\\
&PC := Multiply(Transpose(C), OCCT);\\
&B := Multiply(A, PC);\\
&A1 := Multiply(B, C);\\
&BTB := Multiply(Transpose(B), B);\\
&OBTB := MatrixInverse(BTB);\\
&PB := Multiply(OBTB, Transpose(B));\\
&PsA := Multiply(PC, PB);\\
&\textbf{fi};
\end{aligned}
$$

The program also takes into account the general case where $r(A) \neq m$ and $r(A) \neq n$ and $m-n=1$, in which, unlike the third cycle, matrix $C1$ is formed by sequentially deleting a row equal to m (the number of rows of matrix A) from matrix G and deleting a row equal to n (the number of columns of matrix A) from the resulting matrix [7], [3]:

```
if rA ≠ nA and rA ≠ mA and mA − nA = 1 then G := GaussianElimination(A);
C := DeleteRow(G, mA);
C1 := DeleteRow(C, nA);
rC1 := Rank(C1);
C1C1T := Multiply(C1, Transpose(C1));
OC1C1T := MatrixInverse(C1C1T);
PC1 := Multiply(Transpose(C1), OC1C1T);
B := Multiply(A, PC1);
A1 := Multiply(B, C1);
BTB := Multiply(Transpose(B), B);
OBTB := MatrixInverse(BTB);
PB := Multiply(OBTB, Transpose(B));
PsA := Multiply(PC1, PB);
fi;
```

As we see, the program contains all cases of construction of the skeleton decomposition of the matrix of the system, and as a consequence the construction of the pseudo-matrix of the system. To find the general pseudosolution X_{obns} of the system, we find X_{chns} - the normal pseudosolution of the system, $(E - A^{+} \cdot A) \cdot C = X_{odns}$ - the general pseudosolution of the homogeneous system corresponding to system (1)-(2) and sum up [9], [10]:

```
PsAA := Multiply(PsA, A);
E := IdentityMatrix(3);
EPsAA := MatrixAdd(E, Multiply(PsAA, -1));
MPC := Matrix(3, 1, [C1, C2, C3]);
Xodns := Multiply(EPsAA, MPC);
Xchns := Multiply(PsA, B1);
Xobns := MatrixAdd(Xodns, Xchns);
```

$$Xodns := \begin{bmatrix} \frac{1}{3}\,C1 + \frac{1}{3}\,C2 - \frac{1}{3}\,C3 \\ \frac{1}{3}\,C1 + \frac{1}{3}\,C2 - \frac{1}{3}\,C3 \\ -\frac{1}{3}\,C1 - \frac{1}{3}\,C2 + \frac{1}{3}\,C3 \end{bmatrix}$$

$$Xchns := \begin{bmatrix} 9 \\ 4 \\ 13 \end{bmatrix}$$

$$Xobns := \begin{bmatrix} \frac{1}{3}\,C1 + \frac{1}{3}\,C2 - \frac{1}{3}\,C3 + 9 \\ \frac{1}{3}\,C1 + \frac{1}{3}\,C2 - \frac{1}{3}\,C3 + 4 \\ -\frac{1}{3}\,C1 - \frac{1}{3}\,C2 + \frac{1}{3}\,C3 + 13 \end{bmatrix}$$

As we see, the normal pseudosolutions of the system obtained by the considered methods coincide.

The developed programs for finding pseudosolutions of systems of linear algebraic equations are automated and can be successfully used in scientific research. The advantages of the programs include minimal time consumption of their application, efficiency and solution accuracy.

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