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Article



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CONSTRUCTING OF SECOND-ORDER LINEAR DIFFERENTIAL EQUATION IN MAPLE SYSTEM

Abstract: The linear differential equation theory occupies an important place in mathematics applications to various branches of science and technology, with the fact that by means of these equations the construction and investigation of solutions is greatly facilitated. The rapid advancement of modern systems of analytical calculations creates the possibility of theoretical research and practical implementation in these systems. This is furthered by the great possibilities of computer mathematics systems, which have such advantages as the speed of computation, the efficiency of the result, the reduction of labor intensity of calculations.

Key words: linear differential equations, linearly independent functions, fundamental system of solutions, Wronskian, Lagrange method.

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Introduction

Among the most extensive ways to investigate chemical, physical, economic, biological, and social phenomena through mathematical methods is to model them with differential equations and their systems. The most ordinary models of phenomena and processes can be described by common differential equations and indeed by their more narrow class, the linear differential equations. These equations have remarkable properties that greatly simplify the process of constructing and investigating solutions [1]. At the same time, as the order of a differential equation increases, different problem solving procedure

involving these equations is accompanied by complicated mathematical calculations and computations. This process can be simplified by computer mathematical systems (CMS). Given the advantages that CMSs have, the development of mathematical products in symbolic computation systems is the most a reasonable method for the solution of linear differential equations for their consequent application.

It is well known from the differential equation theory that a linear differential equation has the following form:

$$y^n + p_1(x)y^{(n-1)} + p_2(x)y^{(n-2)} + \dots + p_{n-1}(x)y' + p_n(x)y = f(x), \quad (1.1)$$

where $p_0(x), p_1(x), \dots, p_n(x)$ are coefficients, $f(x)$ are continuous functions in the interval (a, b) has a single solution satisfying the original terms:

$$y = y_0, y' = y_0', \dots, y^{(n-1)} = y_0^{(n-1)},$$

where $x = x_0, x_0 \in (a, b), y_0, y_0', \dots, y_0^{(n-1)}$ are any given numbers. This singular solution is defined and n is differentiable in (a, b) . (1.1) has no special solutions. A partial solution is any solution to (1.1) [2].

Taking $f(x) \equiv 0$ in (a, b) the equation (1.1) is homogeneous:

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$$\begin{vmatrix} y_1^{(n)} & y_1^{(n-1)} & \dots & y_1' & y_1 \\ y_2^{(n)} & y_2^{(n-1)} & \dots & y_2' & y_2 \\ \dots & \dots & \dots & \dots & \dots \\ y_n^{(n)} & y_n^{(n-1)} & \dots & y_n' & y_n \\ y^{(n)} & y^{(n-1)} & \dots & y' & y \end{vmatrix} = 0 \quad \text{at } x \in (a, b), \quad (1.7)$$

which is necessary for the cohesion of system (1.6) and equation (1.2). Equality (1.7) can be expressed in the form:

$$\begin{vmatrix} y_1 & y_2 & \dots & y_n & y \\ y_1' & y_2' & \dots & y_n' & y' \\ \dots & \dots & \dots & \dots & \dots \\ y_1^{(n)} & y_2^{(n)} & \dots & y_n^{(n)} & y^{(n)} \end{vmatrix} = 0 \quad (1.8)$$

Using the decomposition of determinant (1.8) by column, namely by the elements of the last column, we obtain the equation. By dividing all the terms of the last equation by $W(y_1, y_2, \dots, y_n)$, we obtain the equation we are looking for [4].

There is also another way of constructing the (1.2) equation, specifically identifying the equation's coefficients using its fundamental system of solutions

[2]. For example, for a linear homogeneous equation of order 2:

$$y'' + p(x)y' + q(x)y = 0, \quad (1.9)$$

if $\{y_1, y_2\}$ is a fundamental system of solutions to the equation, the coefficients $p(x), q(x)$, are defined as follows [3]:

$$p(x) = -\frac{W'(y_1, y_2)}{W(y_1, y_2)}, \quad q(x) = -\frac{y_1''}{y_1} + \frac{y_1' W'(y_1, y_2)}{y_1 W(y_1, y_2)} \quad (1.10)$$

The equation corresponding to $\{y_1, y_2\}$ - the basic solution system of the equation has the form:

$$y'' - \frac{W'(y_1, y_2)}{W(y_1, y_2)} y' + \left(-\frac{y_1''}{y_1} + \frac{y_1' W'(y_1, y_2)}{y_1 W(y_1, y_2)} \right) y = 0 \quad (1.11)$$

Let's compose a differential equation if $\{y_1 = e^x, y_2 = e^{-x}\}$ is the fundamental system of

solutions to the equation. We equate the Wronskian of solutions to zero:

$$\begin{vmatrix} y_1 & y_2 & y \\ y_1' & y_2' & y' \\ y_1'' & y_2'' & y'' \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} e^x & e^{-x} & y \\ e^x & -e^{-x} & y' \\ e^x & e^{-x} & y'' \end{vmatrix} = 0 \quad (1.2)$$

We calculate the determinant on the left side of the last equality:

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$$\begin{vmatrix} e^x & e^{-x} & y \\ e^x & -e^{-x} & y' \\ e^x & e^{-x} & y'' \end{vmatrix} = e^x \cdot e^{-x} \cdot \begin{vmatrix} 1 & 1 & y \\ 1 & -1 & y' \\ 1 & 1 & y'' \end{vmatrix} =$$

$$= y \cdot (-1)^{1+3} \cdot \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} + y' \cdot (-1)^{2+3} \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + y'' \cdot (-1)^{3+3} \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = 2y - 2y''$$

Then equate the value of the left and right parts of equality (1.12):

$$\begin{aligned} -2y'' + 2y &= 0, \\ y'' - y &= 0. \end{aligned}$$

The last equation is the equation we are looking for, corresponding to $\{y_1 = e^x, y_2 = e^{-x}\}$, the fundamental system of solutions to the equation

You can also use the (1.10) formulas to find the coefficients $p(x), q(x)$ of equation(1.9). Let us sequentially calculate the terms included in formulae(1.10):

$$W(e^x, e^{-x}) = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -e^x \cdot e^{-x} - e^x \cdot e^{-x} = -2$$

$$W'(e^x, e^{-x}) = 0.$$

Then:

$$p(x) = \frac{0}{2} = 0,$$

$$q(x) = -\frac{e^x}{e^x} + \frac{e^x \cdot 0}{e^x \cdot (-2)} = -1.$$

The coefficients $p(x), q(x)$ are substituted in the equation (1.9):

$$\begin{aligned} y'' + 0 \cdot y' + (-1)y &= 0. \\ y'' - y &= 0. \end{aligned}$$

We got the desired equation, which coincides with the equation made by the formula (1.11).

The general solution of heterogeneous equation (1.1) is the sum of its partial solution (one) and the general solution of the corresponding homogeneous equation (1.2) [4]:

$$y = y_q + \underbrace{C_1 y_1 + C_2 y_2 + \dots + C_n y_n}_{y_{\text{одод}}} \quad (1.13)$$

where y_1, y_2, \dots, y_n is the fundamental system of solutions of equations (1.2).

To calculate the total solution of the equation (1.1), e use the Lagrange method, and the overall equation solution (1.1) has the exact same form as the general solution of the relevant homogeneous equation (1.2):

$$y = C_1 y_1 + C_2 y_2 + \dots + C_n y_n,$$

but with one condition: arbitrary constants C_1, C_2, \dots, C_n in (1.2) are replaced by continuously differentiable functions from x :

$$y = C_1(x) y_1 + C_2(x) y_2 + \dots + C_n(x) y_n, \quad (1.14)$$

where the set of functions y_1, y_2, \dots, y_n is some fundamental system of solutions of equations (1.2). A system of differential equations is used to determine $C_i(x)$:

$$\begin{cases} C'_1(x)z_1 + C'_2(x)z_2 + \dots + C'_n(x)z_n = 0 \\ C_1(x)z'_1 + C_2(x)z'_2 + \dots + C_n(x)z'_n = 0 \\ \dots\dots\dots \\ C'_1(x)z_1^{(n-2)} + C'_2(x)z_2^{(n-2)} + \dots + C'_n(x)z_n^{(n-2)} = 0 \\ C_1(x)z_1^{(n-1)} + C_2(x)z_2^{(n-1)} + \dots + C_n(x)z_n^{(n-1)} = f(x) \end{cases} \quad (1.15)$$

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The system (1.15) is an algebraic linear inhomogeneous system with respect to the functions $C_i'(x)$. From (1.15) we find $C_i'(x)$:

$$C_i'(x) = \frac{W_{ni}(x)f(x)}{W(x)} \quad (1.16)$$

In (1.16) $W_{ni}(x)$ is the algebraic addition of the elements of the n row of the determinant $W(x)$, and the functions of the relation $\frac{W_{ni}(x)f(x)}{W(x)}$ are continuous in the interval (a, b) . [3]

Knowing $C_i'(x)$ of (1.16) has the form:

$$C_i(x) = \int_{x_0}^x \frac{W_{ni}(x)f(x)}{W(x)} dx + C_i, (i = 1, 2, \dots, n), \quad (1.17)$$

where $C_i = const$, $x_0 \in (a, b)$.

We obtain the expression for y by substituting the values of functions $C_i(x)$ in (1.14):

$$y = \sum_{i=1}^n y_i \int_{x_0}^x \frac{W_{ni}(x)f(x)}{W(x)} dx + \sum_{i=1}^n C_i y_i \quad (1.18)$$

To obtain the solution of the inhomogeneous equation (1.1), the arbitrary constants must be equal to zero: $C_1 = C_2 \dots = C_n = 0$, then:

$$y_u = \sum_{i=1}^n y_i \int_{x_0}^x \frac{W_{ni}(x)f(x)}{W(x)} dx \quad (1.19)$$

If we have an inhomogeneous linear equation of order 2:

$$y'' + p(x)y' + q(x)y = f(x), \quad (1.20)$$

the general solution by formula (1.18) will be written as [3]:

$$y = -y_1 \int_{x_0}^x \frac{y_2 f(x)}{W(x)} dx + y_2 \int_{x_0}^x \frac{y_1 f(x)}{W(x)} dx + C_1 y_1 + C_2 y_2 \quad (1.21)$$

Writing the private solution of equation (1.20) by formula (1.19) is as follows:

$$y_u = -y_1 \int_{x_0}^x \frac{y_2 f(x)}{W(x)} dx + y_2 \int_{x_0}^x \frac{y_1 f(x)}{W(x)} dx, \quad (1.22)$$

under initial conditions:

$$y_1 = 0, y_1' = 0, \text{ при } x = x_0.$$

Let us make up the Wronskian solutions [6]:

Let us consider the problem of constructing a linear homogeneous differential equation Let us implement the solution in Maple. We enter the data:

```
restart; y1 := exp(x); y2 := exp(-x);
dy1 := diff(y1, x); dy2 := diff(y2, x);
d2y1 := diff(y1, x$2); d2y2 := diff(y2, x$2);
W := Matrix(3, 3, [y1, y2, y, dy1, dy2, dy, d2y1, d2y2, d2y]);
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$$dy1 := e^x$$

$$dy2 := -e^{(-x)}$$

$$d2y1 := e^x$$

$$d2y2 := e^{(-x)}$$

$$W := \begin{bmatrix} e^x & e^{(-x)} & y \\ e^x & -e^{(-x)} & dy \\ e^x & e^{(-x)} & d2y \end{bmatrix}$$

Let's calculate the determinant and equate it to zero, thus obtaining an equation, denoting $d2y = y''$, $dy = y'$:

$$DW := \text{simplify}(\text{Determinant}(W)); eq1 := DW = 0;$$

$$DW := -2 d2y + 2 y$$

$$eq1 := -2 d2y + 2 y = 0$$

It is also possible to make a differential equation in another way, by calculating its coefficients using the formulas (1.10) [6]:

$$\begin{aligned} W1 &:= \text{Matrix}(2, 2, [y1, y2, dy1, dy2]); \\ DW1 &:= \text{simplify}(\text{Determinant}(W1)); \\ dDW1 &:= \text{diff}(DW1, x); \\ p &:= -dDW1/DW1; \\ q &:= -(d2y1/y1) + (dy1*dDW1)/(y1*DW1); \end{aligned}$$

$$W1 := \begin{bmatrix} e^x & e^{(-x)} \\ e^x & -e^{(-x)} \end{bmatrix}$$

$$DW1 := -2$$

$$dDW1 := 0$$

$$p := 0$$

$$q := -1$$

$$eq2 := d2y - y = 0$$

Comparing the equation in the form of $eq1$ and $eq2$ we see that it is not necessary to simplify it as in $eq1$.

Let us consider the problem of constructing a linear homogeneous differential equation and finding a general solution of the corresponding inhomogeneous differential equation by the Lagrange method. We need 1) to construct a differential

equation if $\{y_1 = x, y_2 = x^2\}$ is the fundamental system of solutions of the equation; 2) to find a general solution of the linear inhomogeneous equation, knowing that $f(x) = \frac{1}{x}$.

Let us calculate the Wronskian of solutions $W(x, x^2)$:

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$$W(x, x^2) = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = 2x^2 - x^2 = x^2$$

Although functions $y_1 = x, y_2 = x^2$ are linearly independent at $(-\infty, +\infty)$, $W(x, x^2)$ is zero at $x = 0$. We calculate the coefficients $p(x), q(x)$ using the formulas (1.10):

$$p(x) = -\frac{2x}{x^2} = -\frac{2}{x},$$

$$q(x) = -\frac{0}{x} + \frac{1 \cdot 2x}{x \cdot x^2} = \frac{2}{x^2}.$$

$p(x), q(x)$, have discontinuities at the point $x = 0$.

This is understandable, since $W(x, x^2)$ at the same point it is equal to zero. The differential equation has the form:

$$y'' - \frac{2}{x} \cdot y' + \frac{2}{x^2} y = 0.$$

Let us proceed to the general solution of the inhomogeneous linear differential equation:

$$y'' - \frac{2}{x} \cdot y' + \frac{2}{x^2} y = \frac{1}{x}.$$

Let us apply the Lagrange method and write a general solution to the equation:

$$y = C_1(x)y_1 + C_2(x)y_2,$$

where $C_1(x), C_2(x)$ can be found by solving the system [4]:

$$\begin{cases} y_1 C_1(x) + y_2 C_2(x) = 0 \\ y_1 C_1'(x) + y_2 C_2'(x) = f(x) \end{cases}$$

also by formulas:

$$C_1'(x) = \frac{y_1 f(x)}{W(y_1, y_2)}, \quad C_2'(x) = \frac{y_2 f(x)}{W(y_1, y_2)}$$

Let's calculate $C_1(x)$ and $C_2(x)$ by formulas:

$$C_1'(x) = \frac{y_1 f(x)}{W(y_1, y_2)} = \frac{x \cdot \frac{1}{x}}{x^2} = \frac{1}{x^2}, \quad C_1(x) = \int \frac{dx}{x^2} = -\frac{1}{x} + C_1;$$

$$C_2'(x) = \frac{y_2 f(x)}{W(y_1, y_2)} = \frac{x^2 \cdot \frac{1}{x}}{x^2} = \frac{1}{x}, \quad C_2(x) = \int \frac{dx}{x} = \ln x + C_2.$$

The general solution is:

$$y = \left(-\frac{1}{x} + C_1\right)x + (\ln x + C_2)x^2,$$

$$y = -1 + xC_1 + x^2 \ln x + x^2 C_2.$$

Now let's look at the solution of this example in Maple. For the first part of the problem, we use the program we made earlier and obtain the equation [7], [8]

$$eq1 := -\frac{2 dy}{x} + \frac{2 y}{x^2} + d2y = 0$$

Now compare with the result of manual calculations: the equation is correct. For the second part of the example, enter the formulas for finding

$C_1(x)$ and $C_2(x)$ [9],[10]:

$cc1 := y1 * f / DW1; cc2 := y2 * f / DW1;$

$C11 := Int(cc1, x) = int(cc1, x);$

$C22 := Int(cc2, x) = int(cc2, x);$

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$$cc1 := \frac{1}{x^2}$$

$$cc2 := \frac{1}{x}$$

$$C11 := \int \frac{1}{x^2} dx = -\frac{1}{x}$$

$$C22 := \int \frac{1}{x} dx = \ln(x)$$

Here we find a fundamental equation solution

$$y'' - \frac{2}{x} \cdot y' + \frac{2}{x^2} y = \frac{1}{x}:$$

$$Y := (C11 + C1) * y1 + (C22 + C2) * y2;$$

$$Y := x \left(\int \frac{1}{x^2} dx + C1 \right) + x^2 \left(\int \frac{1}{x} dx + C2 \right) = x \left(-\frac{1}{x} + C1 \right) + x^2 (\ln(x) + C2)$$

Conclusion

As can be seen, finding the analytical solving a non-homogeneous linear differential equation by the Lagrange method is successfully implemented in

MAPLE. Initial entry of the problem data and further calculations involving them indicate that the program is automated.

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