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Issue Article

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MULTIPLICATIVE MATRIX DECOMPOSITIONS IN SYSTEMS OF COMPUTER MATHEMATICS

Abstract: *Representations of matrices in the form of matrix decompositions are successfully used in linear algebra for solving various problems. The implementation in systems of computer mathematics the solution of such problems is greatly simplified due to the built-in functions that perform matrix decompositions. This article deals with the solution of systems of linear algebraic equations by LU -decomposition and QR -decomposition in Maple.*

Key words: *matrix decomposition, LU - decomposition, QR - decomposition, matrix transposition, orthogonal decompositions.*

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Introduction

Among the main problems of computational linear algebra is finding solutions to systems of linear algebraic equations. The effectiveness of the solution method depends on the structure of the matrix of the system. The representation of the matrix of the system in the form of multiplicative decompositions, makes it possible to simplify the solution of the system of equations.

Let us consider a method whose basis is the decomposition of the original matrix of the system into the product of triangular matrices or the method LU - decomposition. Although the algorithms of this method are close to those of the Gaussian method, the sequence of calculations may differ.[1]

In terms of the number of operations required to solve a particular solution system, the Gaussian and LU decomposition methods do not have much difference. But most of the calculations when solving with the LU -decomposition method fall on the decomposition stage. The other part is related to solving two triangular systems. By virtue of this fact, when it is necessary to solve a system that differs only in the vector of free terms, we obtain savings in the decomposition operation. Analogously, when a transposed system needs to be solved after the original system, we can use the previous decomposition and only change the order of solving triangular systems by virtue of transposition.[2]

Let the system of linear algebraic equations have the form:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2 \\ .... \\ a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n = b_n \end{cases} \qquad (1)$$

where $x_1, x_2,..., x_n$ - unknowns, a_{ij} - coefficients at unknowns, $b_1, b_2,..., b_n$ - free terms, which in matrix form has the form:

$$A \cdot X = B, \qquad (2)$$

where $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$ - system matrix, $B = \begin{pmatrix} b_1 \\ b_2 \\ \cdots \\ b_n \end{pmatrix}$ - matrix of free terms, $X = \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix}$ - matrix of unknowns.

The following statement is true for A matrix: if the principal minors of A square matrix are nonzero, then this matrix has a LU -decomposition representation:

$$A = L \cdot U , \tag{3}$$

where L - lower triangular matrix, U - upper triangular matrix with unit diagonal

$$L = \begin{pmatrix} l_{11} & 0 & 0 & \dots & 0 \\ l_{21} & l_{22} & 0 & \dots & 0 \\ l_{31} & l_{32} & l_{33} & \dots & 0 \\ \dots & \dots & \dots & l_{n-1 n-1} & 0 \\ l_{n1} & l_{n2} & \dots & l_{n n-1} & l_{nn} \end{pmatrix}, \quad U = \begin{pmatrix} 1 & u_{12} & u_{13} & \dots & u_{1n} \\ 0 & 1 & u_{23} & \dots & u_{2n} \\ 0 & 0 & 1 & \dots & u_{3n} \\ 0 & 0 & 0 & 1 & u_{n-1\ n} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Moreover, if the elements of L diagonal matrix are non-zero, such decomposition is the only one. The A matrix decomposition is performed in n stages [1]. At each j stage, the l_{ij} elements of the subsequent j column of the matrix L are recalculated sequentially according to the formulas [3]:

$$l_{ij} = a_{ij} - \sum_{k=1}^{j-1} l_{ik} u_{kj} \ , \ i = \overline{j,n} . \tag{4}$$

For the elements of the u_{ji} j - -th row of the matrix U the formulas [3] are used

$$u_{ji} = \frac{a_{ji} - \sum_{k=1}^{j-1} l_{jk} u_{ki}}{l_{jj}} \ , \ i = \overline{j+1,n} \tag{5}$$

Having the representation in the form (3), the matrix equation (2) is written as follows:

$$L \cdot U \cdot X = B \ . \tag{6}$$

Denoting the vector of auxiliary variables by Y, the matrix equation (6) has the form of the following system:

$$\begin{cases} L \cdot Y = B \\ U \cdot X = Y \end{cases} . \tag{7}$$

Solving the first equation of the system (7), we calculate the values of the variables at $i = \overline{1,n}$ by the forward formula:

$$y_i = b_i - \sum_{k=1}^{i-1} l_{ik} y_k \ . \tag{8}$$

When solving the second equation of the system (7), the unknowns X at $i = \overline{n,1}$ are found with the backward formula

$$x_i = \frac{1}{u_{ii}} \left(y_i - \sum_{k=i-1}^{n} u_{ik} x_k \right). \tag{9}$$

The described LU -decomposition system can be implemented only if the elements $l_{jj} \neq 0$. In addition, the proximity of these elements to zero can lead to a large loss of accuracy of the calculations. To avoid this, the solution of the system of equations by the method LU - decomposition must be implemented with the choice of the largest element l_{jj} modulo . Therefore, in addition to matrices L and U it is necessary to store the matrix of permutations P. The permutation matrix P is obtained from the unit matrix E by permutation of rows and columns. Then the matrix A in this case has the form[4]:

$$A = P \cdot L \cdot U \tag{10}$$

Among matrix decompositions, orthogonal ones that preserve the norm of the vector play a special role. One of the most important variants of orthogonal decompositions of A matrix is QR a decomposition of the form[1]

$$A = Q \cdot R, \tag{11}$$

where Q - orthogonal matrix, R - upper triangular matrix.

By orthogonalizing the matrix columns and then orthonormalizing them, we construct a QR - decomposition of the matrix. In the same way, applying orthogonal transformations, it is possible to come to this decomposition [1].

Let the columns $a_1, a_2, ..., a_n$ of the matrix $A = (a_1, a_2, ..., a_n)$ be linearly independent.

We orthogonalize the system of vectors $a_1, a_2, ..., a_n$. Then normalize each vector of the obtained vector system. As a result, we come to an orthonormalized system of vectors:

$$\begin{aligned} q_1 &= u_{11}\, a \\ q_2 &= u_{12}\, a_1 + u_{22}\, a_2 \\ &\dots\dots\dots\ \dots\dots\dots\ \dots\dots\dots\dots \\ q_n &= u_{1n}\, a_1 + u_{2n}\, a_2 + \dots + u_{nn}\, a_n \end{aligned} \tag{12}$$

In matrix notation this gives the equality:

$$Q = A \cdot U \tag{13}$$

where $Q = (q_1, q_2, ..., q_n)$- orthogonal matrix, U - upper triangular matrix:

$$U = \begin{pmatrix} u_{11} & u_{12} & \dots & u_{1,n-1} & u_{1n} \\ 0 & u_{22} & \dots & u_{2,n-1} & u_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & u_{nn} \end{pmatrix} \tag{14}$$

From here we obtain QR-decomposition $A = Q \cdot R$ with orthogonal matrix Q and upper triangular matrix $R = U^{-1}$ [3].

The QR-decomposition of the matrix can be constructed by means of rotations and by means of reflections [1].

Knowing the QR -decomposition of the matrix of the system, the matrix equation (2) can be written in the form:

$$Q \cdot R \cdot X = B, \tag{15}$$

multiplying equation (15) from the right by Q^T, we have:

$$\underbrace{Q^T \cdot Q}_{=I} \cdot R \cdot X = Q^T \cdot B, \tag{16}$$

where follows the solution of the system:

$$R \cdot X = Q^T \cdot B. \tag{17}$$

Since the matrix R is triangular, the solution is obtained by the forward formulas[4].

The solution of systems of linear algebraic equations using matrix expansions of the system is greatly simplified in systems of computer mathematics. These systems have built-in functions that perform matrix expansions[5].

So let's need to find a solution to a system of linear equations

$$\begin{cases} 10x_1 + 6x_2 + 2x_3 = 8, \\ 6x_2 - 2x_3 + 2x_4 = 2, \\ 3x_1 + 5x_2 - x_3 - x_4 = 2, \\ 5x_1 + x_2 - 2x_3 + 4x_4 = 7. \end{cases}$$

We solve the system by the LU-decomposition method in the Maple system of computer mathematics. At the first stage of solving the system we perform LU-decomposition for the matrix of the system A [5]:

P, L, U := LUDecomposition(A);

$$P, L, U := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ \frac{3}{10} & -\frac{8}{5} & 1 & 0 \\ 0 & -3 & \frac{55}{32} & 1 \end{bmatrix}, \begin{bmatrix} 10 & 6 & 2 & 0 \\ 0 & -2 & -3 & 4 \\ 0 & 0 & -\frac{32}{5} & \frac{27}{5} \\ 0 & 0 & 0 & \frac{151}{32} \end{bmatrix}.$$

Let us distinguish the matrices of this decomposition:

$P := LUDecomposition(A, output ='P');$
$L := LUDecomposition(A, output ='L');$
$U := LUDecomposition(A, output ='U');$

$$P := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$L := \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ \frac{3}{10} & -\frac{8}{5} & 1 & 0 \\ 0 & -3 & \frac{55}{32} & 1 \end{bmatrix}$$

$$U := \begin{bmatrix} 10 & 6 & 2 & 0 \\ 0 & -2 & -3 & 4 \\ 0 & 0 & -\frac{32}{5} & \frac{27}{5} \\ 0 & 0 & 0 & \frac{151}{32} \end{bmatrix}$$

Let us make sure that the LU-decomposition of the matrix A is correct[6]:

PLU := P.L.U; $Equal(A, P$.L.U$)$;

$$PLU := \begin{bmatrix} 10 & 6 & 2 & 0 \\ 5 & 1 & -2 & 4 \\ 3 & 5 & -1 & -1 \\ 0 & 6 & -2 & 2 \end{bmatrix}$$

true

In the second step, we directly solve the system, the matrix equation of which now has the form:

$$P \cdot L \cdot U \cdot x = B. \qquad (18)$$

Multiply (18) by the transpose matrix P:

$$P^T \cdot P \cdot L \cdot U \cdot x = P^T \cdot B. \qquad (19)$$

Then we have:

$$L \cdot U \cdot x = P_r, \qquad (20)$$

Since L is a lower triangular matrix it is necessary to find P_l, such that $U \cdot x = P_l$ and:

$$L \cdot P_l = P_r. \qquad (21)$$

Since (20), (21) is satisfied, and considering that U - the upper triangular matrix, the solution of the original system X is found from the equation

$$U \cdot x = P_r \quad . \qquad (22)$$

In Maple the described step is performed by the following sequence of operations[7]:

$PtB := Multiply(\text{Transpose}(P), B);$
$LPtB := ForwardSubstitute(L, PtB);$
$x := BackwardSubstitute(U, LPtB);$

$$PtB := \begin{bmatrix} 8 \\ 7 \\ 2 \\ 2 \end{bmatrix}$$

$$LPtB := \begin{bmatrix} 8 \\ 3 \\ \frac{22}{5} \\ \frac{55}{16} \end{bmatrix}$$

$$x := \begin{bmatrix} \frac{117}{151} \\ \frac{10}{151} \\ -\frac{11}{151} \\ \frac{110}{151} \end{bmatrix}$$

In order to verify the accuracy of the solution, let us calculate the remainder[7],[8]:

$\text{Os} := \text{A.x} - \text{B};$

$$Os := \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The solution of the system by QR-decomposition is carried out in the computer mathematics system Maple similarly to the above mentioned LU-decomposition. At the first stage the system matrix is QR decomposed [5]:

$\text{Q, R} := \text{QRDecomposition}(A);$
$Q := QRDecomposition(A, output = \text{'Q'});$
$R := QRDecomposition(A, output = \text{'R'});$
$\text{QR} := Q.\text{R};$
$Equal(\text{A, Q.R});$

$$Q, R := \begin{bmatrix} \frac{5}{67}\sqrt{134} & \frac{1}{37587}\sqrt{25058} & \frac{2498}{9580197}\sqrt{3193399} & \frac{25}{2578627}\sqrt{389372677} \\ \frac{5}{134}\sqrt{134} & -\frac{133}{75174}\sqrt{25058} & -\frac{4549}{9580197}\sqrt{3193399} & \frac{16}{2578627}\sqrt{389372677} \\ \frac{3}{134}\sqrt{134} & \frac{215}{75174}\sqrt{25058} & -\frac{745}{9580197}\sqrt{3193399} & -\frac{110}{2578627}\sqrt{389372677} \\ 0 & \frac{1}{187}\sqrt{25058} & -\frac{373}{3193399}\sqrt{3193399} & \frac{64}{2578627}\sqrt{389372677} \end{bmatrix},$$

$$\begin{bmatrix} \sqrt{134} & \frac{40}{67}\sqrt{134} & \frac{7}{134}\sqrt{134} & \frac{17}{134}\sqrt{134} \\ 0 & \frac{3}{67}\sqrt{25058} & -\frac{749}{75174}\sqrt{25058} & \frac{19}{25058}\sqrt{25058} \\ 0 & 0 & \frac{1}{561}\sqrt{3193399} & -\frac{6563}{3193399}\sqrt{3193399} \\ 0 & 0 & 0 & \frac{2}{17077}\sqrt{389372677} \end{bmatrix}$$

$$Q := \begin{bmatrix} \frac{5}{67}\sqrt{134} & \frac{1}{37587}\sqrt{25058} & \frac{2498}{9580197}\sqrt{3193399} & \frac{25}{2578627}\sqrt{389372677} \\ \frac{5}{134}\sqrt{134} & -\frac{133}{75174}\sqrt{25058} & -\frac{4549}{9580197}\sqrt{3193399} & \frac{16}{2578627}\sqrt{389372677} \\ \frac{3}{134}\sqrt{134} & \frac{215}{75174}\sqrt{25058} & -\frac{745}{9580197}\sqrt{3193399} & -\frac{110}{2578627}\sqrt{389372677} \\ 0 & \frac{1}{187}\sqrt{25058} & -\frac{373}{3193399}\sqrt{3193399} & \frac{64}{2578627}\sqrt{389372677} \end{bmatrix}$$

$$R := \begin{bmatrix} \sqrt{134} & \frac{40}{67}\sqrt{134} & \frac{7}{134}\sqrt{134} & \frac{17}{134}\sqrt{134} \\ 0 & \frac{3}{67}\sqrt{25058} & -\frac{749}{75174}\sqrt{25058} & \frac{19}{25058}\sqrt{25058} \\ 0 & 0 & \frac{1}{561}\sqrt{3193399} & -\frac{6563}{3193399}\sqrt{3193399} \\ 0 & 0 & 0 & \frac{2}{17077}\sqrt{389372677} \end{bmatrix}$$

$$QR := \begin{bmatrix} 10 & 6 & 2 & 0 \\ 5 & 1 & -2 & 4 \\ 3 & 5 & -1 & -1 \\ 0 & 6 & -2 & 2 \end{bmatrix}$$

true

At the stage of solving the system, operations are performed, according to (16), (17), which in Maple is carried out by a sequence of actions[9],[10]:

$\text{QtB} := \mathit{Multiply}(\,\text{Transpose}(Q), B);\ X := \mathit{BackwardSubstitute}(R, QtB);$

$$QtB := \begin{bmatrix} \frac{121}{134}\sqrt{134} \\ \frac{29}{6834}\sqrt{25058} \\ -\frac{1417}{870927}\sqrt{3193399} \\ \frac{220}{2578627}\sqrt{389372677} \end{bmatrix}$$

$$X := \begin{bmatrix} \frac{117}{151} \\ \frac{10}{151} \\ -\frac{11}{151} \\ \frac{110}{151} \end{bmatrix}$$

As we see, the solution of the system by LU decomposition and by QR -decomposition coincide. Noting the advantage of the solution of SLAE LU-decomposition and QR-decomposition, which is expressed only in the use of the matrix of B free terms at the final stage, it is possible to solve SLAE with the same system matrix, but with different matrices of free terms. To this advantage should be added the accuracy of calculations in systems of computer mathematics, which is clearly confirmed for the solution of systems with coefficients in the form of decimal numbers (in Maple floating point numbers), which takes place during the experiments.

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