SOI: 1.1/TAS DOI: 10.15863/TAS
International Scientific Journal
Theoretical & Applied Science
p-ISSN: 2308-4944 (print) e-ISSN: 2409-0085 (online)
Year: 2023 Issue: 05 Volume: 121
Published: 25.05.2023 http://T-Science.org

Issue Article

U.Z. Djumayozov
Samarkand branch of Tashkent University of Information Technologies
Corresponding author
djumayozov@bk.ru

N.F. Eshmanova
Samarkand branch of Tashkent University of Information Technologies
140100 47A Shokhrukh Mirzo street,
Samarkand, Uzbekistan

NUMERICAL SOLUTION OF THREE-DIMENSIONAL BOUNDARY VALUE PROBLEMS OF THE THEORY OF ELASTICITY IN FINITE DEFORMATIONS

***Abstract**: Usually, when determining the safety margins and reliability of structures and their elements, it is sufficient to solve the boundary value problem of the theory of elasticity at small deformations. With the development of innovative technologies and the widespread use of composite materials, the calculation of materials at large deformations is required. Usually, to solve boundary value problems with finite deformations, variation methods based on potential energy are used. The article formulates a three-dimensional boundary value problem of the theory of elasticity in finite deformations for a parallelepiped with different natural and kinematic boundary conditions. The grid equations are constructed by the finite-difference method. To solve nonlinear difference equations, the method of elastic solutions of Ilyushin is used, i.e. first, the elastic (linear) problem is solved, and then the same problem is solved with a new right-hand side, taking into account the nonlinear part of the original equations. In this case, the finite-difference analogue of the linear problem solved with respect to central nodal values is the basis for an iterative process of the Jakobi type with an approximation order of 2. Using the received numerical results, the distribution of displacements and stresses in a parallelepiped is investigated.*

***Key words**: finite deformation, elasticity, displacement, boundary value problem, difference equation.*
***Language**: English*
Citation: Djumayozov, U. Z., & Eshmanova, N. F. (2023). Numerical solution of three-dimensional boundary value problems of the theory of elasticity in finite deformations. *ISJ Theoretical & Applied Science, 05 (121),* 321-330.
Soi: http://s-o-i.org/1.1/TAS-05-121-47 ***Doi***: crossref https://dx.doi.org/10.15863/TAS.2023.05.121.47
***Scopus ASCC**: 2600.*

Introduction

UDC 519.63+539.37 (575.1)

Currently, engineers and researchers have to deal with the tasks of determining the stress-strain state (SSS) of elements of modern structures and mechanisms, which are under conditions of significantly large deformations. Usually, such problems do not seem to be solved analytically, and the use of modern computing technologies is required. The existing packages of applied programs, widely used in many fields of science and technology, allowing to study the SSS of technically important objects, have been developed mainly on the basis of the finite element method. The formulation of boundary value problems for deformable solids in finite deformations is based on the following fundamental works [2,6,7,10,13,15,17]. The numerical solution of problems of the theory of elasticity and plasticity in finite deformations is a problem in the mechanics of a deformable solid. In this area, the following works can be noted [18–20].

In this paper, for solving boundary value problems in finite deformations, a finite-difference method is proposed in combination with the method of successive approximations. The work consists of an introduction, formulation of the problem of the theory of elasticity in finite deformations and finite-difference equations for nonlinear problems of the theory of elasticity, solved by the method of elastic solutions in combination with the iterative method.

In the second section, the boundary value problem of the theory of elasticity in finite deformations is formulated for a rectangular domain with different boundary conditions, from which the well-known equations of the classical theory of elasticity follow in the case of small deformations. In the third section, to compare the results, the boundary value problem of the theory of elasticity at small deformations is considered. The finite-difference method is used to discretized the resulting differential equations for a rectangular area. Problems of a rectangle with clamped sides and free sides are considered. In the fourth section, the boundary value problem of the theory of elasticity in finite deformations is numerically solved. For a two-dimensional problem, the theory of elasticity in finite deformations, discrete equations are compiled by the finite-difference method. The non-linear finite difference equations are solved by the Jacobi type iterative method. The distribution of stresses inside the clamped and free rectangles is investigated.

BOUNDARY VALUE PROBLEM OF NONLINEAR THEORY OF ELASTICITY IN LAGRANGIAN VARIABLES

This section is devoted to the formulation of boundary value problems in the theory of elasticity for finite and small deformations. Statement of the boundary value problem of elasticity in finite deformations based on the Lagrange formulation [2, 7, 9, 13]. At the same time, the Pioli-Kirchhoff stress tensor and the Green-Lagrange strain tensor are used, the Saint-Venant-Kirchhoff model is used [6, 18, 20], which looks similar to the linear law of the theory of elasticity at small deformations.

The boundary value problem of the nonlinear theory of elasticity in the Lagrangian formulation is considered, i.e.

$$x_i(X,t) = X_i + u_i(X,t), \quad x = X \cdot I + u, \qquad (1)$$

where $X_i, x_i -$ material and spatial coordinates before and after deformation, respectively. From relation (1), one can find that

$$\frac{\partial x_i}{\partial X_j} = \delta_{ij} + \frac{\partial u_i}{\partial X_j}, \quad F = I + \nabla_X u, \quad i,j = 1,2,3. \qquad (2)$$

The last expression is the relationship between the deformation gradient F and the displacement gradients $\nabla_X u$, where $I -$ unit tensor. The Green's strain tensor of the first C and the second E has the form i.e. The first C and the second E Green's strain tensors has the form i.e.,

$$C_{ij} = \frac{\partial x_k}{\partial X_i} \cdot \frac{\partial x_k}{\partial X_j}, \quad C = F^T \cdot F, \quad k = 1,2,3. \qquad (3)$$

$$E_{ij} = \frac{1}{2}(C_{ij} - \delta_{ij}), \quad E = \frac{1}{2}(C - I). \qquad (4)$$

taking into account relation (2), the Green's strain tensor E takes the following form

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j}\right). \qquad (5)$$

The first T_{ij} and second S_{ij} Pioli-Kirchhoff tensors are related as follows [2, 5, 16, 17]

$$T_{ij} = (\delta_{ik} + \frac{\partial u_i}{\partial x_k})S_{kj}, \quad T = F \cdot S. \qquad (6)$$

The relationship between the Kirchhoff stress tensor S_{ij} and Green's strain tensor E_{ij} can be represented in the form of Saint-Venant-Kirchhoff, similar to the usual linear relation in small deformations, i.e.

$$S_{ij} = \lambda E_{kk}\delta_{ij} + 2\mu E_{ij}. \qquad (7)$$

Then the boundary value problem of the theory of elasticity in finite deformations consists of:

equilibrium equations

$$\sum_{j=1}^{3} T_{ij,j} + X_i = 0, \quad div\,T + X = 0, \qquad (8)$$

where

$$T_{ij} = \sum_{k=1}^{3}(\delta_{ik} + u_{i,k})S_{kj}, \quad T = F \cdot S. \qquad (9)$$

Saint-Venant-Kirchhoff law

$$S_{ij} = \lambda E_{kk}\delta_{ij} + 2\mu E_{ij}, \quad E_{kk} = E_{11} + E_{22} + E_{33}, \qquad (10)$$

the Cauchy-Green strain tensor

$$E_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} + \sum_{k=1}^{3} u_{k,i}u_{k,j}), \qquad (11)$$

and the corresponding boundary conditions

$$u_i|_{\Sigma_1} = u_i^0, \quad \sum_{j=1}^{3} S_{ij}n_j|_{\Sigma_2} = S_i^0. \qquad (12)$$

where $u_i -$ displacements, $\lambda, \mu -$ Lame elastic constants, $X_i -$ body force, $\delta_{ij} -$ Kronecker symbol, $n_j -$ is the external normal to the surface Σ_2, $S_1, S_2, S_3 -$ are the components of the external load vector.

A BOUNDARY VALUE PROBLEM THE THEORY OF ELASTICITY AT SMALL DEFORMATIONS

Our goal is to numerically solve the nonlinear boundary value problem (8-12) by the finite-difference method. Why first we will consider the boundary value problem for small deformations, i.e. if $|u_{i,k}| \le 1$ then in relations, (11) and (9) the nonlinear terms can be neglected and the relation between the tensor of deformations and displacements becomes linear, and the Pioli Kirchhoff 2-stress tensor S_{ij} becomes equal to the usual stress tensor σ_{ij}. Then equations (8) - (12) take the following form:

equilibrium equation

$$\sigma_{ij,j} + X_i = 0, \qquad (13)$$

Hooke's law

$$\sigma_{ij} = \lambda\theta\delta_{ij} + 2\mu\varepsilon_{ij}, \qquad (14)$$

Cauchy relation

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \qquad (15)$$

and, boundary conditions

$$u_i|_{\Sigma_1} = u_i^0, \quad \sum_{j=1}^{3}\sigma_{ij}n_j|_{\Sigma_2} = S_i^0. \qquad (16)$$

where σ_{ij} – tensor of stresses, ε_{ij} – tensor of deformations, θ – spherical part of tensor of deformations,

X_i – volume force;

Equations (13-16) in expanded form have the form [1, 8, 11, 16, 19]

$$\begin{aligned}
&\frac{\partial\sigma_{11}}{\partial x_1} + \frac{\partial\sigma_{12}}{\partial x_2} + \frac{\partial\sigma_{13}}{\partial x_3} + X_1 = 0,\\
&\frac{\partial\sigma_{21}}{\partial x_1} + \frac{\partial\sigma_{22}}{\partial x_2} + \frac{\partial\sigma_{23}}{\partial x_3} + X_2 = 0,\\
&\frac{\partial\sigma_{31}}{\partial x_1} + \frac{\partial\sigma_{32}}{\partial x_2} + \frac{\partial\sigma_{33}}{\partial x_3} + X_3 = 0.
\end{aligned} \qquad (17)$$

Hooke's law

$$\begin{aligned}
&\sigma_{11} = (\lambda + 2\mu)\varepsilon_{11} + \lambda\varepsilon_{22} + \lambda\varepsilon_{33}, \ \sigma_{22} = \lambda\varepsilon_{11} + (\lambda + 2\mu)\varepsilon_{22} + \lambda\varepsilon_{33},\\
&\sigma_{33} = \lambda\varepsilon_{11} + \lambda\varepsilon_{22} + (\lambda + 2\mu)\varepsilon_{33}, \ \sigma_{12} = 2\mu\varepsilon_{12}, \ \sigma_{13} = 2\mu\varepsilon_{13}, \ \sigma_{23} = 2\mu\varepsilon_{23}.
\end{aligned} \qquad (18)$$

Cauchy relation

$$\begin{aligned}
&\varepsilon_{11} = \frac{\partial u_1}{\partial x_1}, \quad \varepsilon_{22} = \frac{\partial u_2}{\partial x_2}, \quad \varepsilon_{33} = \frac{\partial u_3}{\partial x_3},\\
&\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right), \ \varepsilon_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right), \ \varepsilon_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right).
\end{aligned} \qquad (19)$$

Substituting (19) into (18) and obtained in (17), and taking into account the following renaming of $u_1 = u, \ u_2 = v, \ u_3 = w, \ x_1 = x, \ x_2 = y, \ x_3 = z$, the equilibrium equation (17) we can write with respect to the displacements

$$\begin{cases}
(\lambda + 2\mu)\dfrac{\partial^2 u}{\partial x^2} + \mu\left(\dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}\right) + (\lambda + \mu)\left(\dfrac{\partial^2 v}{\partial x\partial y} + \dfrac{\partial^2 w}{\partial x\partial z}\right) = 0,\\
\mu\left(\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial z^2}\right) + (\lambda + 2\mu)\dfrac{\partial^2 v}{\partial y^2} + (\lambda + \mu)\left(\dfrac{\partial^2 u}{\partial x\partial y} + \dfrac{\partial^2 w}{\partial y\partial z}\right) = 0,\\
\mu\left(\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2}\right) + (\lambda + \mu)\left(\dfrac{\partial^2 u}{\partial x\partial z} + \dfrac{\partial^2 v}{\partial y\partial z}\right) + (\lambda + 2\mu)\dfrac{\partial^2 w}{\partial z^2} = 0,
\end{cases} \qquad (20)$$

with the following boundary conditions

$$u(x,y,z)|_{\Gamma} = u_1^0, \ v(x,y,z)|_{\Gamma} = u_2^0, \ w(x,y,z)|_{\Gamma} = u_3^0. \qquad (21)$$

To construct a finite-difference scheme for a two-dimensional boundary value problem of the theory of elasticity (20-21), dividing the lengths of the sides of a rectangular region l_k by n_k, we can find that $h_k = l_k / n_k$, where $k = 1,2,3$

$$x_i = h_1 \cdot i, \ i = \overline{0,n_1}, \quad y_j = h_2 \cdot j, \ j = \overline{0,n_2}, \quad z_k = h_3 \cdot k, \ k = \overline{0,n_3}.$$

Replacing the derivatives in the boundary value problem (20-21) with difference relations, one can find the basic finite-difference equations [3-5]

$$(\lambda+2\mu)\frac{u_{i+1,j,k}-2u_{i,j,k}+u_{i-1,j,k}}{h_1^2}+\mu\left(\frac{u_{i,j+1,k}-2u_{i,j,k}+u_{i,j-1,k}}{h_2^2}+\frac{u_{i,j,k+1}-2u_{i,j,k}+u_{i,j,k-1}}{h_3^2}\right)+$$
$$+(\lambda+\mu)\left(\frac{v_{i+1,j+1,k}-v_{i-1,j+1,k}-v_{i+1,j-1,k}+v_{i-1,j-1,k}}{4h_1h_2}+\frac{w_{i+1,j,k+1}-w_{i-1,j,k+1}-w_{i+1,j,k-1}+w_{i-1,j,k-1}}{4h_1h_3}\right)=0, \quad (22)$$

$$(\lambda+2\mu)\frac{v_{i,j+1,k}-2v_{i,j,k}+v_{i,j-1,k}}{h_2^2}+\mu\left(\frac{v_{i+1,j,k}-2v_{i,j,k}+v_{i-1,j,k}}{h_1^2}+\frac{v_{i,j,k+1}-2v_{i,j,k}+v_{i,j,k-1}}{h_3^2}\right)+$$
$$(\lambda+\mu)\left(\frac{u_{i+1,j+1,k}-u_{i-1,j+1,k}-u_{i+1,j-1,k}+u_{i-1,j-1,k}}{4h_1h_2}+\frac{w_{i,j+1,k+1}-w_{i,j-1,k+1}-w_{i,j+1,k-1}+w_{i,j-1,k-1}}{4h_2h_3}\right)=0, \quad (23)$$

$$(\lambda+2\mu)\frac{w_{i,j,k+1}-2w_{i,j,k}+w_{i,j,k-1}}{h_3^2}+\mu\left(\frac{w_{i+1,j,k}-2w_{i,j,k}+w_{i-1,j,k}}{h_1^2}+\frac{w_{i,j+1,k}-2w_{i,j,k}+w_{i,j-1,k}}{h_2^2}\right)+$$
$$+(\lambda+\mu)\left(\frac{u_{i+1,j,k+1}-u_{i-1,j,k+1}-u_{i+1,j,k-1}+u_{i-1,j,k-1}}{4h_1h_3}+\frac{v_{i,j+1,k+1}-v_{i,j-1,k+1}-v_{i,j+1,k-1}+v_{i,j-1,k-1}}{4h_1h_2}\right)=0. \quad (24)$$

Boundary conditions (21) with respect to the nodal points have the form

$$u_{ijk}\big|_{\Gamma}=u_1^0,\ v_{ijk}\big|_{\Gamma}=u_2^0,\quad w_{ijk}\big|_{\Gamma}=u_3^0. \quad (25)$$

The finite difference equations for the boundary conditions of the remaining sides can be written in a similar way. Solving equations (22-25) with respect to $u_{i,j,k}, v_{i,j,k}, w_{i,j,k}$, we can construct the following iterative process [8, 9, 15, 17]

$$u_{i,j,k}^{(n+1)}=((\lambda+2\mu)\frac{u_{i+1,j,k}^{(n)}+u_{i-1,j,k}^{(n)}}{h_1^2}+\mu(\frac{u_{i,j+1,k}^{(n)}+u_{i,j-1,k}^{(n)}}{h_2^2}+\frac{u_{i,j,k+1}^{(n)}+u_{i,j,k-1}^{(n)}}{h_3^2})+$$
$$+(\lambda+\mu)(\frac{v_{i+1,j+1,k}^{(n)}-v_{i-1,j+1,k}^{(n)}-v_{i+1,j-1,k}^{(n)}+v_{i-1,j-1,k}^{(n)}}{4h_1h_2}+\frac{w_{i+1,j,k+1}^{(n)}-w_{i-1,j,k+1}^{(n)}-w_{i+1,j,k-1}^{(n)}+w_{i-1,j,k-1}^{(n)}}{4h_1h_3}))/ \quad (26)$$
$$/(\frac{2(\lambda+2\mu)}{h_1^2}+\frac{2\mu}{h_2^2}+\frac{2\mu}{h_3^2}),$$

$$v_{i,j,k}^{(n+1)}=((\lambda+2\mu)\frac{v_{i,j+1,k}^{(n)}+v_{i,j-1,k}^{(n)}}{h_2^2}+\mu(\frac{v_{i+1,j,k}^{(n)}+v_{i-1,j,k}^{(n)}}{h_1^2}+\frac{v_{i,j,k+1}^{(n)}+v_{i,j,k-1}^{(n)}}{h_3^2})+$$
$$+(\lambda+\mu)(\frac{u_{i+1,j+1,k}^{(n)}-u_{i-1,j+1,k}^{(n)}-u_{i+1,j-1,k}^{(n)}+u_{i-1,j-1,k}^{(n)}}{4h_1h_2}+\frac{w_{i,j+1,k+1}^{(n)}-w_{i,j-1,k+1}^{(n)}-w_{i,j+1,k-1}^{(n)}+w_{i,j-1,k-1}^{(n)}}{4h_2h_3}))/ \quad (27)$$
$$/(\frac{2(\lambda+2\mu)}{h_2^2}+\frac{2\mu}{h_1^2}+\frac{2\mu}{h_3^2}),$$

$$w_{i,j,k}^{(n+1)} = ((\lambda+2\mu)\frac{w_{i,j,k+1}^{(n)} + w_{i,j,k-1}^{(n)}}{h_3^2} + \mu(\frac{w_{i+1,j,k}^{(n)} + w_{i-1,j,k}^{(n)}}{h_1^2} + \frac{w_{i,j+1,k}^{(n)} + w_{i,j-1,k}^{(n)}}{h_2^2}) +$$
$$+(\lambda+\mu)(\frac{u_{i+1,j,k+1}^{(n)} - u_{i-1,j,k+1}^{(n)} - u_{i+1,j,k-1}^{(n)} + u_{i-1,j,k-1}^{(n)}}{4h_1h_3} + \frac{v_{i,j+1,k+1}^{(n)} - v_{i,j-1,k+1}^{(n)} - v_{i,j+1,k-1}^{(n)} + v_{i,j-1,k-1}^{(n)}}{4h_1h_2}))/ \quad (28)$$
$$/(\frac{2(\lambda+2\mu)}{h_3^2} + \frac{2\mu}{h_1^2} + \frac{2\mu}{h_3^2}).$$

Equations (26-28) constitute an iterative process of Jacobi type, and its convergence is ensured by the general convergence theorem for iterative processes [10, 14, 19, 21]. Note that the conditions for diagonal dominance for equations (26-28) are satisfied.

Let a parallelepiped with edges of length $l_i, i=1,2,3$ be clamped along all faces [11, 12, 14], i.e.

$$u_{0,j,k} = \frac{1}{10}\sin(\frac{y_j\pi}{h_2})\sin(\frac{z_k\pi}{h_3}),\ u_{n_1,j,k} = -\frac{1}{10}\sin(\frac{y_j\pi}{h_2})\sin(\frac{z_k\pi}{h_3}),$$
$$v_{i,0,k} = \frac{1}{10}\sin(\frac{x_i\pi}{h_1})\sin(\frac{z_k\pi}{h_3}),\ v_{i,n_2,k} = -\frac{1}{10}\sin(\frac{x_i\pi}{h_1})\sin(\frac{z_k\pi}{h_3}),$$
$$w_{i,j,0} = \frac{1}{10}\sin(\frac{x_i\pi}{h_1})\sin(\frac{y_j\pi}{h_2}),\ w_{i,j,n_3} = -\frac{1}{10}\sin(\frac{x_i\pi}{h_1})\sin(\frac{y_j\pi}{h_2}), \quad (29)$$
$$v_{0,j,k} = 0,\ v_{n_1,j,k} = 0,\ w_{0,j,k} = 0,\ v_{n_1,j,k} = 0,$$
$$u_{i,0,k} = 0,\ u_{i,n_2,k} = 0,\ w_{i,0,k} = 0,\ w_{i,n_2,k} = 0,$$
$$u_{i,j,0} = 0,\ u_{i,j,n_3} = 0,\ v_{i,j,0} = 0,\ v_{i,j,n_3} = 0.$$

Equations (26-28) were solved with the following initial data

$$\nu = \frac{1}{3},\quad E = 2\cdot 10^4,\quad l_i = 1,\quad i=1,2,3,\quad n_1 = n_2 = n_3 = 10.$$

The iterative process continues until the following inequality is satisfied $\left\|\vec{u}^{(k+1)} - \vec{u}^{(k)}\right\| \le \varepsilon$. It took 75 iterations to solve the equations with an accuracy of $\varepsilon = 0.001$.

TABLE 1. Displacement values *u(x,y,z)* for *z*=0.7

	x=0	*x=0.1*	*x=0.2*	*x=0.3*	*x=0.4*	*x=0.5*	*x=0.6*	*x=0.7*	*x=0.8*	*x=0.9*	*x=1*
y=0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
y=0.1	0.0250	0.0119	0.0046	0.0012	0.0000	0.0000	0.0000	-0.0012	-0.0046	-0.0119	-0.0250
y=0.2	0.0476	0.0274	0.0138	0.0060	0.0020	0.0000	-0.0020	-0.0060	-0.0138	-0.0274	-0.0476
y=0.3	0.0655	0.0406	0.0227	0.0111	0.0044	0.0000	-0.0044	-0.0111	-0.0227	-0.0406	-0.0655
y=0.4	0.0769	0.0494	0.0287	0.0148	0.0061	0.0000	-0.0061	-0.0148	-0.0287	-0.0494	-0.0769
y=0.5	0.0809	0.0524	0.0308	0.0161	0.0067	0.0000	-0.0067	-0.0161	-0.0308	-0.0524	-0.0809
y=0.6	0.0769	0.0494	0.0287	0.0148	0.0061	0.0000	-0.0061	-0.0148	-0.0287	-0.0494	-0.0769
y=0.7	0.0655	0.0406	0.0227	0.0111	0.0044	0.0000	-0.0044	-0.0111	-0.0227	-0.0406	-0.0655
y=0.8	0.0476	0.0274	0.0138	0.0060	0.0020	0.0000	-0.0020	-0.0060	-0.0138	-0.0274	-0.0476
y=0.9	0.0250	0.0119	0.0046	0.0012	0.0000	0.0000	0.0000	-0.0012	-0.0046	-0.0119	-0.0250
y=1	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Impact Factor:

ISRA (India)	= **6.317**	**SIS** (USA)	= **0.912**	**ICV** (Poland)	= **6.630**
ISI (Dubai, UAE)	= **1.582**	**РИНЦ** (Russia)	= **3.939**	**PIF** (India)	= **1.940**
GIF (Australia)	= **0.564**	**ESJI** (KZ)	= **8.771**	**IBI** (India)	= **4.260**
JIF	= **1.500**	**SJIF** (Morocco)	= **7.184**	**OAJI** (USA)	= **0.350**

TABLE 2. Displacement values *v(x,y,z)* for *y*=0.9

	x=0	*x=0.1*	*x=0.2*	*x=0.3*	*x=0.4*	*x=0.5*	*x=0.6*	*x=0.7*	*x=0.8*	*x=0.9*	*x=1*
y=0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
y=0.1	0.0000	-0.0024	-0.0074	-0.0119	-0.0148	-0.0159	-0.0148	-0.0019	-0.0074	-0.0024	0.0000
y=0.2	0.0000	-0.0074	-0.0180	-0.0274	-0.0335	-0.0356	-0.0335	-0.0274	-0.0180	-0.0074	0.0000
y=0.3	0.0000	-0.0119	-0.0274	-0.0406	-0.0494	-0.0524	-0.0494	-0.0406	-0.0274	-0.0119	0.0000
y=0.4	0.0000	-0.0148	-0.0335	-0.0494	-0.0598	-0.0634	-0.0598	-0.0494	-0.0335	-0.0148	0.0000
y=0.5	0.0000	-0.0159	-0.0356	-0.0524	-0.0634	-0.0672	-0.0634	-0.0524	-0.0356	-0.0159	0.0000
y=0.6	0.0000	-0.0148	-0.0335	-0.0494	-0.0598	-0.0634	-0.0598	-0.0494	-0.0335	-0.0148	0.0000
y=0.7	0.0000	-0.0119	-0.0274	-0.0406	-0.0494	-0.0524	-0.0494	-0.0406	-0.0274	-0.0119	0.0000
y=0.8	0.0000	-0.0074	-0.0180	-0.0274	-0.0335	-0.0356	-0.0335	-0.0274	-0.0180	-0.0074	0.0000
y=0.9	0.0000	-0.0024	-0.0074	-0.0119	-0.0148	-0.0159	-0.0148	-0.0019	-0.0074	-0.0024	0.0000
y=1	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

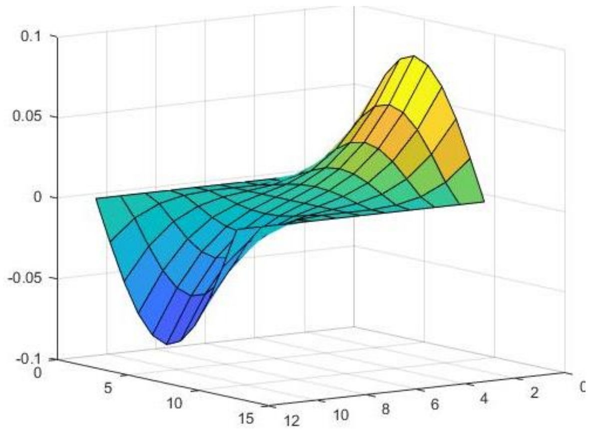

FIGURE 1. The graph of the displacement distribution distribution *u(x,y,z)* for *z*=0.7

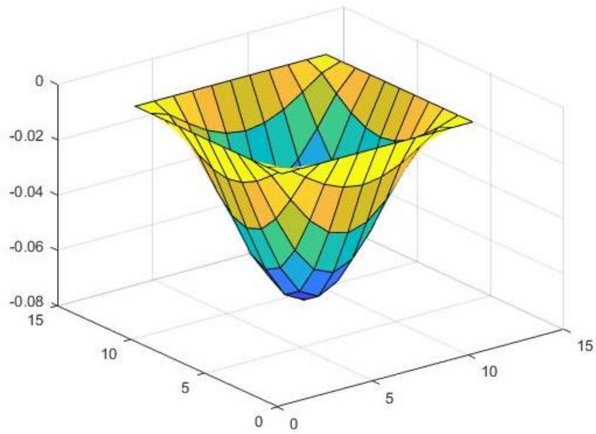

FIGURE 2. The graph of the displacement *u(x,y,z)* for *y*=0.9

THE BOUNDARY VALUE PROBLEM OF THE THEORY OF ELASTICITY IN FINITE DEFORMATIONS

This section is devoted to the numerical solution of the elasticity boundary value problem in finite deformations (8-12). First, we write these equations in expanded form. Then the equilibrium equation has the following form

$$\frac{\partial T_{11}}{\partial x}+\frac{\partial T_{12}}{\partial y}+\frac{\partial T_{13}}{\partial z}+X_1=0,$$
$$\frac{\partial T_{21}}{\partial x}+\frac{\partial T_{22}}{\partial y}+\frac{\partial T_{23}}{\partial z}+X_2=0, \qquad (30)$$
$$\frac{\partial T_{31}}{\partial x}+\frac{\partial T_{32}}{\partial y}+\frac{\partial T_{33}}{\partial z}+X_3=0,$$

where T_{ij} is the first Pioli Kirchhoff stress tensor and which related with the second Pioli Kirchhoff stress tensor as

$$T_{11}=(1+\frac{\partial u}{\partial x})\sigma_{11}+\frac{\partial u}{\partial y}\sigma_{21}+\frac{\partial u}{\partial z}\sigma_{31},\ T_{22}=(1+\frac{\partial v}{\partial y})\sigma_{22}+\frac{\partial v}{\partial x}\sigma_{12}+\frac{\partial v}{\partial z}\sigma_{32},$$
$$T_{33}=(1+\frac{\partial w}{\partial z})\sigma_{33}+\frac{\partial w}{\partial x}\sigma_{13}+\frac{\partial w}{\partial y}\sigma_{23},\ T_{12}=(1+\frac{\partial u}{\partial x})\sigma_{12}+\frac{\partial u}{\partial y}\sigma_{22}+\frac{\partial u}{\partial z}\sigma_{32},\quad (31)$$
$$T_{21}=(1+\frac{\partial v}{\partial y})\sigma_{21}+\frac{\partial v}{\partial x}\sigma_{11}+\frac{\partial v}{\partial z}\sigma_{31},\ T_{23}=(1+\frac{\partial v}{\partial y})\sigma_{23}+\frac{\partial v}{\partial x}\sigma_{13}+\frac{\partial v}{\partial z}\sigma_{33},$$
$$T_{32}=(1+\frac{\partial w}{\partial z})\sigma_{32}+\frac{\partial w}{\partial x}\sigma_{12}+\frac{\partial w}{\partial y}\sigma_{22}.$$

Hooke's law

$$\sigma_{11}=(\lambda+2\mu)\varepsilon_{11}+\lambda(\varepsilon_{22}+\varepsilon_{33}),\ \sigma_{22}=(\lambda+2\mu)\varepsilon_{22}+\lambda(\varepsilon_{11}+\varepsilon_{33}),$$
$$\sigma_{33}=(\lambda+2\mu)\varepsilon_{33}+\lambda(\varepsilon_{11}+\varepsilon_{22}),\ \sigma_{12}=2\mu\varepsilon_{12},\ \sigma_{13}=2\mu\varepsilon_{13},\ \sigma_{23}=2\mu\varepsilon_{23}.\quad (32)$$

and nonlinear geometric relationship has following form

$$\varepsilon_{11}=\frac{\partial u}{\partial x}+\frac{1}{2}(\frac{\partial u}{\partial x})^2+\frac{1}{2}(\frac{\partial v}{\partial x})^2+\frac{1}{2}(\frac{\partial w}{\partial x})^2,\ \varepsilon_{22}=\frac{\partial v}{\partial y}+\frac{1}{2}(\frac{\partial u}{\partial y})^2+\frac{1}{2}(\frac{\partial v}{\partial y})^2+\frac{1}{2}(\frac{\partial w}{\partial y})^2,$$
$$\varepsilon_{33}=\frac{\partial w}{\partial z}+\frac{1}{2}(\frac{\partial u}{\partial z})^2+\frac{1}{2}(\frac{\partial v}{\partial z})^2+\frac{1}{2}(\frac{\partial w}{\partial z})^2,\ \varepsilon_{12}=\frac{1}{2}(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}+\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}),\quad (33)$$
$$\varepsilon_{13}=\frac{1}{2}(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}+\frac{\partial u}{\partial x}\frac{\partial u}{\partial z}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial z}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial z}),\ \varepsilon_{23}=\frac{1}{2}(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}+\frac{\partial u}{\partial y}\frac{\partial u}{\partial z}+\frac{\partial v}{\partial y}\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\frac{\partial w}{\partial z}).$$

Substituting (33) into (32), and (32) into (31) from (30), we can find the equilibrium equations for the displacements

$$(\lambda+2\mu+A_1)\frac{\partial^2 u}{\partial x^2}+(\lambda+\mu+D_1)\frac{\partial^2 v}{\partial x\partial y}+(\lambda+\mu+F_1)\frac{\partial^2 w}{\partial x\partial y}+(\mu+B_1)\frac{\partial^2 u}{\partial y^2}+(\mu+C_1)\frac{\partial^2 u}{\partial z^2}+N_1=0,$$
$$(\lambda+2\mu+A_2)\frac{\partial^2 v}{\partial y^2}+(\lambda+\mu+D_2)\frac{\partial^2 u}{\partial x\partial y}+(\lambda+\mu+F_2)\frac{\partial^2 w}{\partial y\partial z}+(\mu+B_2)\frac{\partial^2 v}{\partial x^2}+(\mu+C_2)\frac{\partial^2 v}{\partial z^2}+N_2=0,\quad (34)$$
$$(\lambda+2\mu+A_3)\frac{\partial^2 w}{\partial z^2}+(\lambda+\mu+D_3)\frac{\partial^2 u}{\partial x\partial z}+(\lambda+\mu+F_3)\frac{\partial^2 v}{\partial y\partial z}+(\mu+B_3)\frac{\partial^2 w}{\partial x^2}+(\mu+C_3)\frac{\partial^2 w}{\partial y^2}+N_3=0.$$

The introduced designations A_1, B_1, C_1, D_1, F_1, N_1, A_2, B_2, C_2, D_2, F_2, N_2, A_3, B_3, C_3, D_3, F_3, N_3 in equation (34) represent the nonlinear parts of the equations and due to the cumbersomeness the expressions of these notations are not given in the work. Notice, that if they are neglected, the equations of the elasticity theory at small deformations follow from them. Boundary conditions recorded relative to displacements has the following form

$$u(x,y,z)|_{\Gamma}=u_1^0,\ v(x,y,z)|_{\Gamma}=u_2^0,\ w(x,y,z)|_{\Gamma}=u_3^0.\quad (35)$$

The finite difference analogue of equations (34) and (35) has the form

$$(\lambda+2\mu+A_1)\frac{u_{i+1,j,k}-2u_{i,j,k}+u_{i-1,j,k}}{h_1^2}+(\mu+B_1)(\frac{u_{i,j+1,k}-2u_{i,j,k}+u_{i,j-1,k}}{h_2^2})+(\mu+C_1)(\frac{u_{i,j,k+1}-2u_{i,j,k}+u_{i,j,k-1}}{h_3^2})+$$
$$+(\lambda+\mu+D_1)(\frac{v_{i+1,j+1,k}-v_{i-1,j+1,k}-v_{i+1,j-1,k}+v_{i-1,j-1,k}}{4h_1h_2})+(\lambda+\mu+F_1)(\frac{w_{i+1,j,k+1}-w_{i-1,j,k+1}-w_{i+1,j,k-1}+w_{i-1,j,k-1}}{4h_1h_3})+N_1=0,\quad (36)$$

$$(\lambda+2\mu+A_2)\frac{v_{i,j+1,k}-2v_{i,j,k}+v_{i,j-1,k}}{h_2^2}+(\mu+B_2)(\frac{v_{i+1,j,k}-2v_{i,j,k}+v_{i-1,j,k}}{h_1^2})+(\mu+C_2)(\frac{v_{i,j,k+1}-2v_{i,j,k}+v_{i,j,k-1}}{h_3^2})+$$
$$+(\lambda+\mu+D_2)(\frac{u_{i+1,j+1,k}-u_{i-1,j+1,k}-u_{i+1,j-1,k}+u_{i-1,j-1,k}}{4h_1h_2})+(\lambda+\mu+F_2)(\frac{w_{i,j+1,k+1}-w_{i,j-1,k+1}-w_{i,j+1,k-1}+w_{i,j-1,k-1}}{4h_2h_3})+N_2=0, \quad (37)$$

$$(\lambda+2\mu+A_3)\frac{w_{i,j,k+1}-2w_{i,j,k}+w_{i,j,k-1}}{h_3^2}+(\mu+B_3)(\frac{w_{i+1,j,k}-2w_{i,j,k}+w_{i-1,j,k}}{h_1^2})+(\mu+C_3)(\frac{w_{i,j+1,k}-2w_{i,j,k}+w_{i,j-1,k}}{h_2^2})+$$
$$+(\lambda+\mu+D_3)(\frac{u_{i+1,j,k+1}-u_{i-1,j,k+1}-u_{i+1,j,k-1}+u_{i-1,j,k-1}}{4h_1h_3})+(\lambda+\mu+F_3)(\frac{v_{i,j+1,k+1}-v_{i,j-1,k+1}-v_{i,j+1,k-1}+v_{i,j-1,k-1}}{4h_1h_2})+N_3=0, \quad (38)$$

boundary conditions

$$u_{ijk}\Big|_{\Gamma}=u_1^0,\ v_{ijk}\Big|_{\Gamma}=u_2^0,\quad w_{ijk}\Big|_{\Gamma}=u_3^0. \qquad (39)$$

Similar to the previous section, equations (36-39) may be resolved with respect to $u_{i,j,k}, v_{i,j,k}, w_{i,j,k}$ for organize iterative process with boundary conditions (35). The initial data was taken as in the previous task. Tables 3 and 4 show the displacement values in different sections of the parallelepiped. Using the tables, you can make sure that the results are symmetrical, which corresponds to the boundary conditions.

TABLE 3. Displacement values u(x,y,z) for z=0,7

	x=0	*x=0.1*	*x=0.2*	*x=0.3*	*x=0.4*	*x=0.5*	*x=0.6*	*x=0.7*	*x=0.8*	*x=0.9*	*x=1*
y=0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
y=0.1	0.0250	0.0106	0.0039	0.0008	-0.0002	0.0000	0.0002	-0.0008	-0.0039	-0.0106	-0.0250
y=0.2	0.0476	0.0261	0.0131	0.0056	0.0019	0.0000	-0.0019	-0.0056	-0.0131	-0.0261	-0.0476
y=0.3	0.0655	0.0392	0.0217	0.0107	0.0042	0.0000	-0.0042	-0.0107	-0.0217	-0.0392	-0.0655
y=0.4	0.0769	0.0478	0.0276	0.0143	0.0059	0.0000	-0.0059	-0.0143	-0.0276	-0.0478	-0.0769
y=0.5	0.0809	0.0508	0.0296	0.0156	0.0065	0.0000	-0.0065	-0.0156	-0.0296	-0.0508	-0.0809
y=0.6	0.0769	0.0478	0.0276	0.0143	0.0059	0.0000	-0.0059	-0.0143	-0.0276	-0.0478	-0.0769
y=0.7	0.0655	0.0392	0.0217	0.0107	0.0042	0.0000	-0.0042	-0.0107	-0.0217	-0.0392	-0.0655
y=0.8	0.0476	0.0261	0.0131	0.0056	0.0019	0.0000	-0.0019	-0.0056	-0.0131	-0.0261	-0.0476
y=0.9	0.0250	0.0106	0.0039	0.0008	-0.0002	0.0000	0.0002	-0.0008	-0.0039	-0.0106	-0.0250
y=1	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

TABLE 4. Displacement values *v(x,y,z)* for *y*=0.9

	x=0	*x=0.1*	*x=0.2*	*x=0.3*	*x=0.4*	*x=0.5*	*x=0.6*	*x=0.7*	*x=0.8*	*x=0.9*	*x=1*
y=0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
y=0.1	0.0000	-0.0021	-0.0067	-0.0106	-0.0132	-0.0141	-0.0132	-0.0106	-0.0067	-0.0021	0.0000
y=0.2	0.0000	-0.0067	-0.0171	-0.0261	-0.0319	-0.0339	-0.0319	-0.0261	-0.0171	-0.0067	0.0000
y=0.3	0.0000	-0.0106	-0.0261	-0.0392	-0.0478	-0.0508	-0.0478	-0.0392	-0.0261	-0.0106	0.0000
y=0.4	0.0000	-0.0132	-0.0319	-0.0478	-0.0585	-0.0623	-0.0585	-0.0478	-0.0319	-0.0132	0.0000
y=0.5	0.0000	-0.0141	-0.0339	-0.0508	-0.0623	-0.0664	-0.0623	-0.0508	-0.0339	-0.0141	0.0000
y=0.6	0.0000	-0.0132	-0.0319	-0.0478	-0.0585	-0.0623	-0.0585	-0.0478	-0.0319	-0.0132	0.0000
y=0.7	0.0000	-0.0106	-0.0261	-0.0392	-0.0478	-0.0508	-0.0478	-0.0392	-0.0261	-0.0106	0.0000
y=0.8	0.0000	-0.0067	-0.0171	-0.0261	-0.0319	-0.0339	-0.0319	-0.0261	-0.0171	-0.0067	0.0000
y=0.9	0.0000	-0.0021	-0.0067	-0.0106	-0.0132	-0.0141	-0.0132	-0.0106	-0.0067	-0.0021	0.0000
y=1	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Figures 3 and 4 show and compare the changes in displacements *u(x,y,z)* and *v(x,y,z)* constructed at some points of the parallelepiped by solving boundary value problems of the theory of elasticity at finite and small deformations. Comparisons show that the curves are almost the same, indicating that the nonlinear parts are insignificant for solids. The effects of nonlinear terms become noticeable for rubber-like materials.

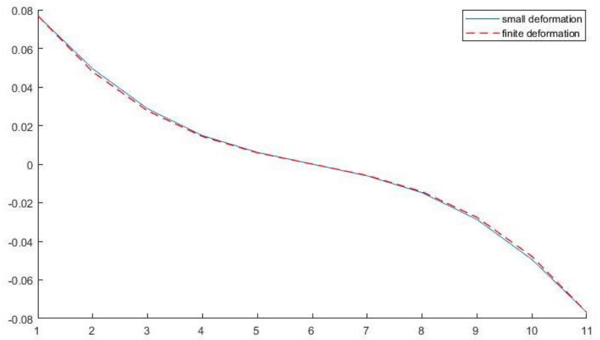

FIGURE 3. Changes in the displacement of *u(x,y,z)* along *x* at the nodal point *y=0.6, z=0.7*

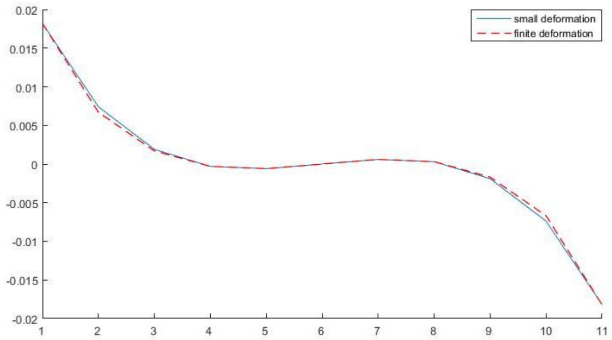

FIGURE 4. Changes in the displacement of *v(x,y,z)* along *y* at the nodal point *x=0.9, z=0.2*

CONCLUSION

In this paper the boundary value problems of the theory of elasticity in finite deformations for a parallelepiped are formulated. The parallelepiped was considered with natural and kinematic boundary conditions. For solving the three -dimensional boundary value problem of elasticity theory in finite deformations the finite-difference method is used. Using the received numerical results, the distribution of displacements and stresses in a parallelepiped is investigated. The results also are compared with a results of elasticity parallelepiped in small deformations. The comparison shows that for solid materials as steel and copper influence of the finite deformation is negligible. However, composites widely used in practice show large deformations. The finite-difference method, applied in this work for solving boundary value problems of the elasticity theory under finite deformations in combination with the method of elastic solutions can be used to solve spatial elastoplastic boundary value problems in finite deformations.

References:

1. Pobedrya, B.E. (1995). *Numerical Methods in the Theory of Elasticity and Plasticity*. (p.523). Moscow: Izd-vo MGU, (1st ed.) 1981; (2nd ed.) 1995) [in Russian].
2. Kolarov, D., Baltov, A., & Boncheva, N. (1979). *Mechanics of Plastic Media* [Russian translation]. (p.411). Moscow: Mir.
3. Li, S., Gupta, A., & Markenscoff, X. (2015). Conservation Laws of Linear Elasticity in Stress Formulations. *Proceedings: Mathematical, Physical and Engineering Sciences.* Vol. 461, No. 2013 (Jan., 2015), 99-116.
4. Filonenko-Borodich, M. (2003). *Theory of Elasticity. University Press of the Pacific* (November 6, 2003), 396 p.
5. Muravleva, L. V. (1986). *Application of Variation Method for Space Problem of Elasticity in Stresses.* Candidate's Dissertation in Mathematical Physics. Mosk. Gos. Univ., (p.352). Moscow.
6. Ike, C.C. (2018). On Maxwell's Stress Functions for Solving Three Dimensional Elasticity Problems in the Theory of Elasticity. *JCAMECH* Vol. 49, No. 2, December 2018, 342-350. DOI: 10.22059/JCAMECH.2018.266787.330.
7. Khaldjigitov, A.A., Yusupov, Y.S., Khudazarov, R.S., & Sadullaeva, D.A. (2019). On the Thermoplasticity Constitutive Relations for Isotropic and Transversely Isotropic Materials. *International Journal of Mechanical and Production Engineering Research and Development* (IJMPERD), 2019, Vol.9, Issue 4, pp. 467-478.
8. Green, A. (1965). *Large Elastic Deformations and Nonlinear Mechanics of Continuous Medium*. (p.456). Moscow: Mir.
9. Khaldjigitov, A.A., Djumayozov, U.Z., & Ibodulloev, Sh.R. (n.d.). Effective Finite-Difference Method for Elastoplastic Boundary Value Problems. *AIP Conference Proceedings*, 2365(1): 020008. https://doi.org/10.1063/5.0057047.
10. Korobeynikov, S.N. (n.d.). *Nonlinear Deformation of Solids.* (p.262). Novosibirsk: Publishing house of the SB RAS.

11. Lurie, A.I. (2018). *Nonlinear Theory of Elasticity*. (p.512). Moscow: Science.
12. Auden, D. (1976). *Finite Elements in Nonlinear Mechanics of Continuous Media*. (p.464). Moscow:Mir.
13. Golovanov, A.I., & Sagdatullin, M.K. (2010). Nonlinear Problem of Hyperelastic Deformation of a Polylinear Finite Element of a Shell of Average Thickness. News of higher educational institutions. Volga region. *Physical and mathematical sciences*, No.4 (16), pp. 39-49.
14. Papadopoulos, J.L. (2018). A General Framework for the Numerical Solution of Problems in Finite Elasto-Plasticity. *Computer Methods in Applied Mechanics and Engineering*, 159 (2018), pp.1-18.
15. Reese, S., & Wriggers, P. (2015). A Finite Element Method for Stability Problems in Finite Elasticity. *International Journal for Numerical Methods in Engineering*, vol-38, 17, pp.1171-1200.
16. Chamberland, E., Fortin, A., & Fortin, M. (2010). Comparison of the Performance of some Finite Element Discretization for Large Deformation Elasticity Problems. *Computers and Structures*, 2010.
17. Novatsky, V. (1975). *The Theory of Elasticity*. (p.872). Moscow: Mir.
18. Samarski, A.A., & Nikolaev, E.S. (1978). *Methods for Solving Grid Equations*. (p.592). Moscow: «Science».
19. Khaldjigitov, A., Djumayozov, U., & Tilovov, O. (2023). A new approach to numerical simulation of boundary value problems of the theory of elasticity in stresses and strains. *EUREKA*: *Physics and Engineering*, 2, 00-00. doi: http://doi.org/10.21303/2461-4262.2023.002735.
20. Khaldjigitov, A.A., & Djumayozov, U.Z. (2022). Numerical Solution of the Two-Dimensional Elasticity Problem in Strains. *Mathematics and Statistics*, №5, Vol 10, 2022, pp.1081-1088. DOI: 10.13189/ms.2022.100518.
21. Dzhumajozov, U.Z. (2022). Chislennoe reshenie ploskoj kraevoj zadachi teorii uprugosti v deformacijah. *Problemy vychislitel`noj i prikladnoj matematiki*, №3(41), 120-129.
22. Eidel, B., & Gruttmann, F. (2003). Elastoplastic orthotropy at finite strains: multiplicative formulation and numerical implementation. *Computational Materials Science*, Vol 28, No 3-4, pp.732-742. doi: 10.1016/j.commatsci.2003.08.027.
23. Schroder, J., Gruttmann, F., & Loblein, J. (2021). A simple orthotropic finite elasto-plasticity model based on generalized stress-strain stress-strain measures. *Computational Mechanics*. 2021, Vol 30, No 1. pp. 48-64. doi: 10.1007/s00466-002-0366-3.