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SOI: [1.1/TAS](#) DOI: [10.15863/TAS](#)

International Scientific Journal Theoretical & Applied Science

p-ISSN: 2308-4944 (print) e-ISSN: 2409-0085 (online)

Year: 2023 Issue: 09 Volume: 125

Published: 03.09.2023 <http://T-Science.org>

Issue

Article



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NUMERICAL SOLUTION OF THE HOMOGENEOUS DIRICHLET PROBLEM FOR THE HELMHOLTZ EQUATION USING MATLAB

Abstract: In this paper, we describe the numerical solution of the homogeneous Dirichlet problem for the Helmholtz equation. It shows what difficulties arose and how they were solved. This may be useful for researchers solving similar problems.

Key words: the Dirichlet problem, the Helmholtz operator, numerical methods, MATLAB, FDM, FEM.

Language: English

Citation: Kozhevnikov, V. A., & Biryukov, A. M. (2023). Numerical solution of the homogeneous Dirichlet problem for the Helmholtz equation using MATLAB. *ISJ Theoretical & Applied Science*, 09 (125), 166-170.

Soi: <http://s-o-i.org/1.1/TAS-09-125-12> **Doi:**  <https://dx.doi.org/10.15863/TAS.2023.09.125.12>

Scopus ASCC: 2600.

Introduction

The homogeneous Dirichlet problem for the Helmholtz equation is considered in the description of many physical phenomena. In particular, the gain coefficient averaged over the cross section of the active element and the output power of the He-Ne laser radiation are expressed through the solution of the corresponding problem. These issues are considered in detail in a series of articles [1-14]. In [1], a method was proposed for finding an approximate solution of the Dirichlet problem for the Helmholtz equation (1)-(2):

$$\Delta f + \lambda^2 f = 0 \quad (1)$$

$$f|_{\partial\Omega} = 0 \quad (2)$$

where $\Omega \subset \mathbb{R}^2$ - bounded simply connected domain with boundary $\partial\Omega$. This method has been tested for domains Ω that admit an analytical solution (circle, rectangle and ellipse - respectively, in cylindrical, Cartesian and elliptical coordinates), and then, using this method, the value of the quantity E (see formula (3) below) was studied for various regions Ω . The value of E , commonly referred to as the average-to-

peak ratio, is defined as follows. Let f be the solution of the homogeneous Dirichlet problem (1)-(2) corresponding to the first (minimal) eigenvalue of the Laplace operator λ_{\min} , then:

$$E = \frac{1}{|\Omega| f_{\max}} \int_{\Omega} f(S) dS \quad (3)$$

where $|\Omega|$ is the Lebesgue measure of the domain Ω , $f_{\max} := \max_{\Omega} f$. The search for the optimal cross-sectional shape of the He-Ne laser in terms of gain is reduced to the search for the region Ω , for which the value of E is maximum. Finding such a simply connected region is still an unsolved problem, the best estimates for the value of E are obtained in [15-16].

It was decided to compare the method proposed in [1] with the methods of finite differences (FDM) and finite elements (FEM) widely used in numerical solutions of the Helmholtz equation.

Numerical calculations

To be sure what comparison is reliable same computing environment (MATLAB) were used, programs for suggested method and FDM were

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written by authors and for FEM already existing library Partial Differential Equation Toolbox [17] was used.

Let us describe what difficulties we had in solving problem (1)-(2) using the FDM and FEM methods.

The first issue is common to both FEM and FDM. Since the equation is homogeneous and the boundary conditions are zero, the systems of equations formulated in both methods become homogeneous, which leads to the difficulty of obtaining an exact numerical solution.

After studying the available materials, no conventionally accepted solution to this problem was found. Despite the logical progression of the steps, which arise from the theoretical description of the method, a qualitative practical implementation has yet to be discovered.

Therefore, we proposed our own solution, and later it was discovered that when using this solution, the calculation results converge within an acceptable margin of error with theoretical calculations and other methods.

The essence of the solution lies in introducing a function $g(x, y) = f(x, y) + a$, where 'a' is an arbitrary constant. By substituting $g(x, y)$ instead of $f(x, y)$ into the equation (1), we obtain the same equation for g , but with a right-hand side of the form $\lambda^2 h^2 a$, where h is the distance between nodes, and with border conditions equal to a .

Let's provide the derivation of the of the resulting equation. From (1) in the Cartesian coordinate system

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \lambda^2 f = 0$$

Let's approximate second-order derivatives.

$$\frac{f(x-h, y) - 2f(x, y) + f(x+h, y)}{h^2} + \frac{f(x, y-h) - 2f(x, y) + f(x, y+h)}{h^2} + \lambda^2 f(x, y) = 0$$

From here we get:

$$f(x-h, y) + f(x+h, y) + f(x, y-h) + f(x, y+h) + \lambda^2 h^2 f(x, y) - 4f(x, y) = 0$$

Because

$$g(x, y) = f(x, y) + a$$

$$f(x, y) = g(x, y) - a$$

then we get:

$$g(x-h, y) - a + g(x+h, y) - a + g(x, y-h) - a + g(x, y+h) - a - 4g(x, y) + 4a + \lambda^2 h^2 g(x, y) - \lambda^2 h^2 a = 0$$

And finally we obtain equations for g :

$$g(x-h, y) + g(x+h, y) + g(x, y-h) + g(x, y+h) + (\lambda^2 h^2 - 4)g(x, y) = \lambda^2 h^2 a \quad (4)$$

The second problem pertains solely to the finite difference method. The issue arises due to the presence of second-order derivatives with respect to both variables in the equation, which prevents the traditional layer-by-layer computation of the entire calculation domain based on initial conditions using a "sweep" method typically employed in many cases of this method's application. To circumvent this, all nodes within the calculation domain are linearly numbered layer by layer. For instance, if a node has the number 'x', the node directly above it will have the number 'x+s', where 's' is the layer width; the node below it will have the number 'x-s', and the nodes to the left and right will be 'x-1' and 'x+1', respectively.

For each node, an equation (4) is formulated. To achieve this, a sparse matrix of size $N \cdot N$ is created, where N is the number of nodes in the calculation domain. Each row of the matrix contains coefficients of the equation corresponding to the node with the respective number. The columns represent the values of the function at each node (considered as variables).

Nodes outside the calculation domain are assigned a value of a in the column corresponding to the node number. Then, the matrix equation $AX=B$ is solved, where A is the aforementioned matrix, X is the vector of function values at the nodes within the calculation domain, and B is the vector of right-hand side values of the equations (for nodes inside the domain, this is $\lambda^2 h^2 a$, where h is the distance between nodes; for nodes outside the domain, this is a).

To determine whether a node lies within the shape or not, its coordinates are checked against the boundary equation. The built-in MATLAB function is employed to solve this equation.

Let's explain the difference between common and our cases by images.

On the Figure 1 the initial calculation domain is depicted, with green nodes representing nodes where the function's values are defined by initial conditions.

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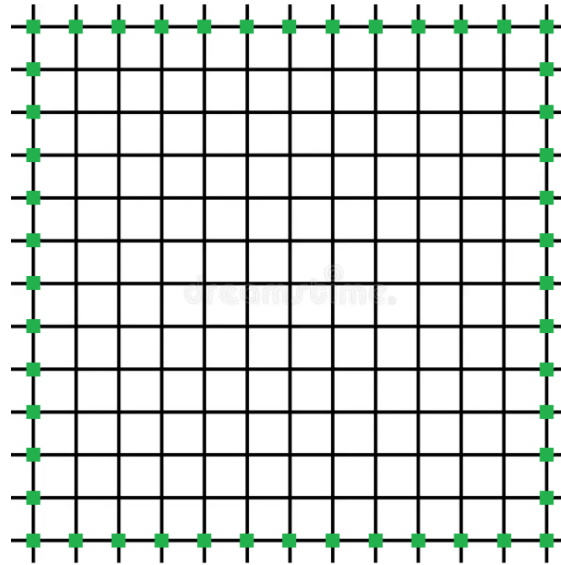


Fig. 1. Beginning state

Figure 2 depicts the initial step of the sweep. Blue nodes are the ones from which calculations are

made, while the red node is the one being calculated (notice that all blue nodes are pre-determined).

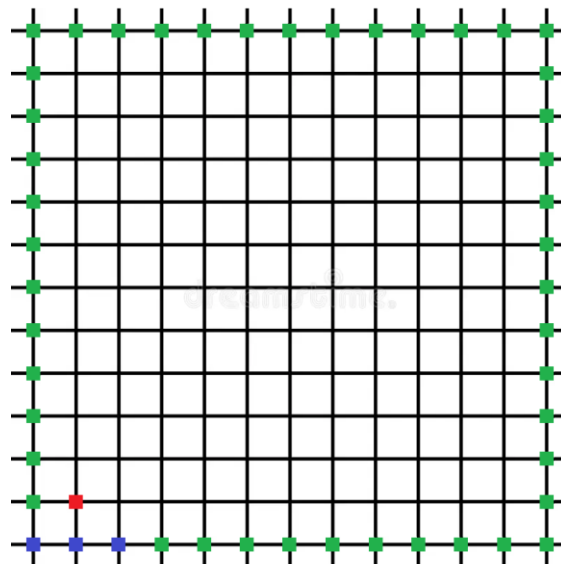


Fig. 2. Sweep

In Figure 3, the moment is shown when the sweep has already computed two layers. The orange nodes represent the nodes that have been calculated.

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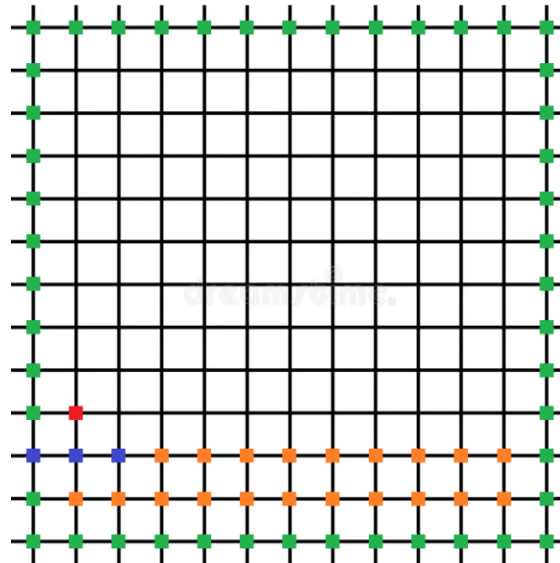


Fig. 3. Sweep 2.

Figure 4 illustrates our scenario (as evident, not all blue nodes are known).

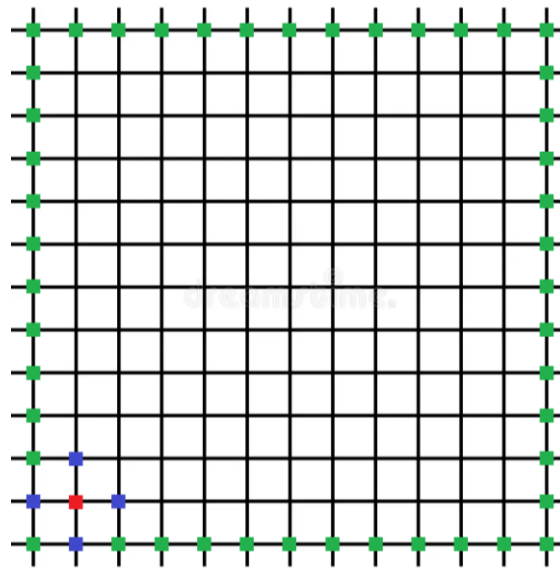


Fig 4. Our case

Conclusion

Calculations were carried out both by the method proposed in [1] and by the methods of FDM and FEM - for different domains Ω and a different number of calculation nodes. The calculation results will be published later. According to the results of the calculations, it can be seen that the method proposed in [1] shows approximately the same fairly high

accuracy as the FDM method, and with the same number of nodes, the accuracy of these methods is higher than that of the FEM. With the same number of calculation nodes, the method proposed in [1] is faster than FDM and FEM. Currently, research is being carried out to find the optimal shape of the region Ω , which gives the maximum value of E from equality (3).

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