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Issue **Article**

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THE BEST APPROXIMATIONS OF RANDOM FIELDS IN ROOT-MEAN SQUARE AND UNIFORM METRIC

Abstract: *In the paper, we study the asymptotically best approximations of random fields by linear positive operators in the mean square and uniform metrics.*

Key words: *random field, r.f., linear positive operator, l.p.o., approximation, best constant, continuity module.*

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Introduction

Denote by $\overline{C_\Omega}(R^2)$ the class of all real, measurable r.f.'s $\mathfrak{z}(t,s)$ uniformly continuous in the mean square (m.s.), defined on a probability space $(\Omega, \mathfrak{F}, P)$, $M|\mathfrak{z}(t,s)|^2 \le C$, $0 < C < \infty$, $(t,s) \in R^2$

The function

$$\omega_{\mathfrak{z}}^{(1)}(x_1,x_2) = \max_{\substack{|t-t'|\le x_1 \\ |s-s'|\le x_2 \\ -\infty<t,s<\infty}} \{M|\mathfrak{z}(t,s) - \mathfrak{z}(t',s')|^2\}^{\frac{1}{2}},$$

$$x_1, x_2 \ge 0,$$

is said to be the continuity module of the first type ([1], [6]) of a r.f. $\mathfrak{z}(t,s) \in \overline{C_\Omega}(R^2)$.

We call the function

$$\omega_{\mathfrak{z}}^{(2)}(x) = \max_{(t-t')^2+(s-s')^2 \le x^2} \{M|\mathfrak{z}(t,s) - \mathfrak{z}(t',s')|^2\}^{\frac{1}{2}},$$

$$x \ge 0$$

the continuity module of the second type of a r.f. $\mathfrak{z}(t,s) \in \overline{C_\Omega}(R^2)$.

The continuity modules $\omega_{\mathfrak{z}}^{(1)}(x_1,x_2)$ and $\omega_{\mathfrak{z}}^{(2)}(x)$ of $\mathfrak{z}(t,s) \in \overline{C_\Omega}(R^2)$ have the following properties:

1^0. For any $0 \le x_1 \le x'_1$, $0 \le x_2 \le x'_2$, the inequalities:

$$\omega_{\mathfrak{z}}^{(1)}(x_1, x_2) \le \omega_{\mathfrak{z}}^{(1)}(x'_1, x_2) \le \omega_{\mathfrak{z}}^{(1)}(x'_1, x'_2)$$

hold.

2^0. For any $n \in N$ and $0 \le x_1 \le x_2$,

$$\omega_{\mathfrak{z}}^{(1)}(n x_1, n x_2) \le n\omega_{\mathfrak{z}}^{(1)}(x_1, x_2).$$

3^0. For any

$$0 \le x_1 \le x_2, \quad \omega_{\mathfrak{z}}^{(2)}(x_1) \le \omega_{\mathfrak{z}}^{(2)}(x_2).$$

4^0. For any $n \in N$,

$$0 \le x_1 \le x_2, \quad \omega_{\mathfrak{z}}^{(2)}(nx) \le n\omega_{\mathfrak{z}}^{(1)}(x).$$

5^0 . $\omega_{\mathfrak{z}}^{(1)}(\frac{\sqrt{2}}{2}x, \frac{\sqrt{2}}{2}x) \le \omega_{\mathfrak{z}}^{(2)}(x) \le \omega_{\mathfrak{z}}^{(1)}(x,x) \le \omega_{\mathfrak{z}}^{(2)}(x\sqrt{2})$, $x \ge 0$.

Let $(X_k, Y_k)_{k=1}^{\infty}$ be a sequence of independent identically distributed real random vectors with a joint distribution function $F_{t,s}(x,y)$ depending on the parameters t and s such that (X_1,Y_1) has the mathematical expectation (t,s) and covariation matrix $\begin{pmatrix}\sigma_1(t) & 0\\ 0 & \sigma_2(s)\end{pmatrix}$, where the parameter (t,s) changes on the set $Q\subset R^2$.

Consider the approximation of $\mathfrak{z}(t,s) \in \overline{C_\Omega}(R^2)$ on the compact set $G \subset Q$ by the l.p.o.

$$P_n(\mathfrak{z};t,s)=\int_{R^2}\mathfrak{z}(x,y)dF_{t,s}^{(n)}(x,y) \qquad (1)$$

Where $F_{t,s}^{(n)}(x,y) = P\{\frac{S_n^{(1)}}{n} < x ; \frac{S_n^{(2)}}{n} < y\}$,

$$S_n^{(1)} = \sum_{k=1}^{n} X_k, \qquad S_n^{(2)} = \sum_{k=1}^{n} Y_k,$$

the integral in (1) is understood in the m.s. sense.

Note that the asymptotically best approximations of random processes by l.p.o. are studied in work [13].

According to [5, p. 268], r.f. $P_n(\mathfrak{z};t,s)$ is defined for all

$(t,s) \in Q$.

The results of [1], [6] imply that

$$\max_{(t,s)\in G}\{ M|\mathfrak{z}(t,s) - P_n(\mathfrak{z};t,s)|^2\}^{\frac{1}{2}} \le C_1\, \omega_{\mathfrak{z}}^{(1)}(\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}})$$

and

$$\max_{(t,s)\in G}\{ M|\mathfrak{z}(t,s) - P_n(\mathfrak{z};t,s)|^2\}^{\frac{1}{2}} \le C_2\, \omega_{\mathfrak{z}}^{(2)}(\frac{1}{\sqrt{n}})$$

The smallest ("best", "optimal") constants that can be put instead of the constants C_1 and C_2 on the right sides of these inequalities,

$$C_1 = C_1(F) = \sup_{\substack{\mathfrak{z}\in\overline{C_\Omega}(R^2)\\ n\in N}} \{\frac{\max_{(t,s)\in G}\{ M|\mathfrak{z}(t,s)-P_n(\mathfrak{z};t,s)|^2\}^{\frac{1}{2}}}{\omega_{\mathfrak{z}}^{(1)}(\frac{1}{\sqrt{n}},\frac{1}{\sqrt{n}})}\}$$

and

$$C_2 = C_2(F) = \sup_{\substack{\mathfrak{z}\in\overline{C_\Omega}(R^2)\\ n\in N}} \left\{\frac{\max_{(t,s)\in G}\{ M|\mathfrak{z}(t,s)-P_n(\mathfrak{z};t,s)|^2\}^{\frac{1}{2}}}{\omega_{\mathfrak{z}}^{(2)}(\frac{1}{\sqrt{n}})}\right\},$$

respectively.

The study of the exact values of the smallest constants C_1 and C_2 leads to complex calculations related to the specifics of the distribution $F_t^{(n)}(x)$. Instead of them, we present in the work asymptotically optimal constants, i.e.

$$C_1^* = \overline{\lim_{n\to\infty}}\, [\sup_{\mathfrak{z}\in\overline{C_\Omega}(R^2)} \{\frac{\max_{(t,s)\in G}\{ M|\mathfrak{z}(t,s)-P_n(\mathfrak{z};t,s)|^2\}^{\frac{1}{2}}}{\omega_{\mathfrak{z}}^{(1)}(\frac{1}{\sqrt{n}},\frac{1}{\sqrt{n}})}\}] \text{ and}$$

$$C_2^* = \overline{\lim_{n\to\infty}}\, [\sup_{\mathfrak{z}\in\overline{C_\Omega}(R^2)} \{\frac{\max_{(t,s)\in G}\{ M|\mathfrak{z}(t,s)-P_n(\mathfrak{z};t,s)|^2\}^{\frac{1}{2}}}{\omega_{\mathfrak{z}}^{(2)}(\frac{1}{\sqrt{n}})}\}]$$

Introduce the following notations:

$$\sigma_1 = \sup_{(t,s)\in G}\{\sigma_1(t)\}, \qquad \sigma_2 = \sup_{(t,s)\in G}\{\sigma_2(s)\},$$

$$D_k = D_k(\sigma_1, \sigma_2) = \{(u,v); |u| \le \frac{k}{\sigma_1}, |v| \le \frac{k}{\sigma_2}\},$$

$$æ_1 = æ_1(\sigma_1, \sigma_2) = \sum_{k=0}^{\infty}\int_{R^2\setminus D_k} d\Phi(u,v), \text{ where}$$

$$\Phi(u,v) = \frac{1}{2\pi}\int_{-\infty}^{u}\int_{-\infty}^{v} exp\{-\frac{x^2+y^2}{2}\}dxdy.$$

$$Q_k = Q_k(\sigma_1, \sigma_2) = \{(u,v): k^2 \le \sigma_1^2u^2 + \sigma_2^2v^2 < (k+1)^2\},$$

$$æ_2 = æ_2(\sigma_1, \sigma_2) = \sum_{k=0}^{\infty}\int_{R^2\setminus Q_k} d\Phi(u,v),$$

$$\Phi(u,v) = \frac{1}{2\pi}\int_{-\infty}^{u}\int_{-\infty}^{v}\exp\{-\frac{x^2+y^2}{2}\}dxdy$$

Let $G \subset Q$ be any compact set, and the following conditions are satisfied:

(A_1): $\sup_{(t,s)\in G} M|X_1 - t|^6 \le L_1$, $\sup_{(t,s)\in G} M|Y_1 - s|^6 \le L_2$,

$0 < L_i < \infty, i = 1,2$.

(B_1): $\sigma_1 \equiv \sigma_1(t_0) = \sup_{(t,s)\in G}\sigma_1(t) > 0$,

$\sigma_2 \equiv \sigma_2(s_0) = \sup_{(t,s)\in G}\sigma_2(s) > 0$

(C_1): the set G is such that $(t_0, s_0) \in G$.

Theorem 1. If conditions (A_1), (B_1), (C_1) are satisfied, then

a) for any $\mathfrak{z}(t,s) \in \overline{C_\Omega}(R^2)$ and $\varepsilon > 0$, there exists $n_0(\varepsilon) \in N$ such that for all $n > n_0(\varepsilon)$, the following relation holds:

$$\max_{(t,s)\in G}\{ M|\mathfrak{z}(t,s) - P_n(\mathfrak{z};t,s)|^2\}^{\frac{1}{2}} \le$$

$$\le [æ_1(\sigma_1,\sigma_2) + \varepsilon]\omega_{\mathfrak{z}}^{(1)}(\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}})$$

b) this inequality is unimprovable for the class $\overline{C_\Omega}(R^2)$ in the sense that for any $\varepsilon > 0$, there exist $n_1(\varepsilon) \in N$ and a r.f. $\mathfrak{z}n(t,s) \in \overline{C_\Omega}(R^1)$ such that for all $n > n_1(\varepsilon)$, the inequality

$$\max_{(t,s)\in G}\{ M|\mathfrak{z}(t,s) - P_n(\mathfrak{z}n;t,s)|^2\}^{\frac{1}{2}} >$$

$$> [æ_1(\sigma_1,\sigma_2) - \varepsilon]\, \omega_{\mathfrak{z}n}(\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}) \quad \text{holds.}$$

Theorem 2. If conditions (A_1), (B_1), (C_1) are satisfied, then the relation

$$\overline{\lim_{n\to\infty}}\ \sup_{\mathfrak{z}\in\overline{C_\Omega}(R^2)}\{\frac{\max_{(t,s)\in G}[M|\mathfrak{z}(t,s)-Pn(\mathfrak{z};t,s)|^2]^{\frac{1}{2}}}{\omega_{\mathfrak{z}}^{(1)}(\frac{1}{\sqrt{n}},\frac{1}{\sqrt{n}})}\} =$$

$= æ_1(\sigma_1, \sigma_2)$ holds.

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Theorem 3. If conditions (A_1), (B_1), (C_1) are satisfied, then

a) for any $\mathfrak{z}(t,s) \in \overline{C_\Omega}(R^2)$ and $\varepsilon > 0$, there exists $n_0(\varepsilon) \in N$ such that for all $n > n_0(\varepsilon)$, the following relation holds:

$$\max_{t\in G}\{ M|\mathfrak{z}(t,s) - P_n(\mathfrak{z};t,s)|^2\}^{\frac{1}{2}} \le$$

$$\le [\text{æ}_2(\sigma_1,\sigma_2) + +\varepsilon]\omega_{\mathfrak{z}}^{(2)}\left(\frac{1}{\sqrt{n}}\right)$$

b) this inequality is unimprovable for the class $\overline{C_\Omega}(R^2)$ in the sense that for any $\varepsilon > 0$, there exist $n_1(\varepsilon) \in N$ and a r.f. $\mathfrak{z}n(t,s) \in \overline{C_\Omega}(R^1)$ such that for all $n > n_1(\varepsilon)$, the inequality

$$\max_{(t,s)\in G}\{ M|\mathfrak{z}(t,s) - P_n(\mathfrak{z}n;t,s)|^2\}^{\frac{1}{2}} >$$

$$> [\text{æ}_2(\sigma_1,\sigma_2) - \varepsilon]\,\omega_{\mathfrak{z}n}^{(2)}\left(\frac{1}{\sqrt{n}}\right)$$

holds.

Theorem 4. Let conditions (A_1), (B_1), (C_1) be satisfied. Then the following relation holds:

$$\overline{\lim_{n\to\infty}}\ \sup_{\mathfrak{z}\in\overline{C_\Omega}(R^2)}\left\{\frac{\max_{(t,s)\in G}\{ M|\mathfrak{z}(t,s)-P_n(\mathfrak{z};t,s)|^2\}^{\frac{1}{2}}}{\omega_{\mathfrak{z}}^{(2)}\left(\frac{1}{\sqrt{n}}\right)}\right\} =$$

$$= \text{æ}_2(\sigma_1,\sigma_2).$$

Consider the examples.

Let $\{(X_k,Y_k)\}_{k=1}^{\infty}$ be a sequence of independent identically distributed random vectors generating a l.p.o. $P_n(\mathfrak{z},t,s)$.

Example 1. Let (X_1,Y_1) be a vector with independent components such that X_1 and Y_1 have a Bernoulli distribution with parameters t and s, respectively. Then the set or parameters Q=$[0,1]^2$ and the operator

$$P_n(\mathfrak{z},t,s) = B_n(\mathfrak{z},t,s) =$$
$$= \sum_{k=0,}^{n}\sum_{e=0}^{n} C_n^k C_n^e t^k s^e \cdot (1 - -t)^{n-k}(1-s)^{n-e}\cdot \mathfrak{z}\left(\frac{k}{n};\frac{e}{n}\right)$$

is the Bernstein polynomial of two variables. If we take G=Q, then the asymptotically optimal constant in the estimate

$$\max_{(t,s)\in[0,1]^2}\{ M[\mathfrak{z}(t,s) - B_n(\mathfrak{z},t,s)]^2\}^{\frac{1}{2}} \le$$

$$\le C_1\omega_{\mathfrak{z}}^{(1)}\left(\frac{1}{\sqrt{n}};\frac{1}{\sqrt{n}}\right)$$

is $C_1^* = \text{æ}_1(\frac{1}{2},\frac{1}{2})$, and in the estimate

$$\max_{(t,s)\in[0,1]^2}\{ M[\mathfrak{z}(t,s) - B_n(\mathfrak{z},t,s)]^2\}^{\frac{1}{2}} \le C_2\omega_{\mathfrak{z}}^{(2)}\left(\frac{1}{\sqrt{n}}\right),$$

the asymptotically optimal constant is $C_2^* = \text{æ}_2(\frac{1}{2},\frac{1}{2})$.

Example 2. Let (X_1,Y_1) be a vector with uncorrelated components that have a Bernoulli distribution with parameters t and s, respectively, and a joint distribution:

$P\{X_1=1,Y_1=1\} = 0$, $P\{X_1=1,Y_1=0\} = t$,

$P\{X_1=0,Y_1=1\} = s$, $P\{X_1=0, Y_1=0\} = 1-t-s$.

Then

$$P_n(\mathfrak{z},t,s) = \sum_{0\le k+e\le n} C_{n-e}^k C_n^e s^k (1-t-s)^{n-e-k}\,\mathfrak{z}\left(\frac{k}{n};\frac{e}{n}\right)$$

is a Bernstein-type polynomial introduced by Lorentz [12].

In this case the set of parameters Q={(t,s): $0 \le t+s \le 1$}.

If we assume that G=Q, then the asymptotically optimal constant in the estimate

$$\max_{(t,s)\in[0,1]^2}\{ M[\mathfrak{z}(t,s) - P_n(\mathfrak{z},t,s)]^2\}^{\frac{1}{2}}$$

$$\le C_1\omega_{\mathfrak{z}}^{(1)}\left(\frac{1}{\sqrt{n}};\frac{1}{\sqrt{n}}\right)$$

is $C_1^* = \text{æ}_1(\frac{1}{2},\frac{1}{2})$, and in the estimate

$$\max_{(t,s)\in[0,1]^2}\{ M[\mathfrak{z}(t,s) - P_n(\mathfrak{z},t,s)]^2\}^{\frac{1}{2}} \le C_2\omega_{\mathfrak{z}}^{(2)}\left(\frac{1}{\sqrt{n}}\right),$$

the asymptotically optimal constant is

$C_2^* = \text{æ}_2(\frac{1}{2},\frac{1}{2})$.

Example 3. Let (X_1,Y_1) be a vector with uncorrelated components that have a Poisson distribution with parameters t and s, respectively. Then

$$P_n(\mathfrak{z},t,s) = \sum_{k=0}^{\infty}\sum_{e=0}^{\infty}\frac{(nt)^k(ns)^e}{k!\cdot e!}exp\{-n(t+s)\}\cdot \mathfrak{z}\left(\frac{k}{n};\frac{e}{n}\right)$$

is the Mirakyan operator [7] for random fields and the set of parameters Q = $[0,\infty)^2$.

If G = $[0,1]^2$, Then the asymptotically optimal constant in the estimate

$$\max_{(t,s)\in[0,1]^2}\{ M[\mathfrak{z}(t,s) - P_n(\mathfrak{z},t,s)]^2\}^{\frac{1}{2}}$$

$$\le C_1\omega_{\mathfrak{z}}^{(1)}\left(\frac{1}{\sqrt{n}};\frac{1}{\sqrt{n}}\right)$$

is $C_1^* = \text{æ}_1(1,1)$, and in the estimate

$$\max_{(t,s)\in[0,1]^2}\{ M[\mathfrak{z}(t,s) - P_n(\mathfrak{z},t,s)]^2\}^{\frac{1}{2}} \le C_2\omega_{\mathfrak{z}}^{(2)}\left(\frac{1}{\sqrt{n}}\right)$$

the asymptotically optimal constant is $C_2^* = \text{æ}_2(1,1)$.

In the deterministic case, i.e. when $\mathfrak{z}(t,s)$ is a non-random function, Theorems 1 – 4 imply the following results:

Let $\overline{C}(R^2)$ be the class of all uniformly continuous functions bounded on R^2. Consider the approximation of the function f (t,s) $\in \overline{C}(R^2)$ on a compact set G $\subset Q$ by the l.p.o.

$P_n(f;t,s) = \int_{-\infty}^{\infty} f(x,y)dF_{t,s}^{(n)}(x,y)$.

Denote by

$$\omega_f^{(1)}(x,y) = \sup_{\substack{|t-t'|\le x\\ |s-s'|\le y\\ -\infty<t,s<\infty}} |f(t,s) - f(t',s')|$$

the continuity module of the first kind, by

$$\omega_f^{(2)}(\delta) = \sup_{\substack{\sqrt{(t-t')^2+(s-s')^2}\le\delta\\ -\infty<t,s<\infty}} |f(t,s) - f(t',s')|$$

the continuity module of the second kind of a function $f(x,y)$ [11].

Theorem 5. If conditions (A_1), (B_1), (C_1) are satisfied, then

a) for any $f(t,s) \in \overline{C}(R^2)$ and $\varepsilon > 0$, there exists $n_0(\varepsilon) \in N$ such that for all $n > n_0(\varepsilon)$, the relation

$$\max_{(t,s)\in G} |f(t,s) - P_n(f;t,s)| \le$$
$$\le [\,æ_1(\sigma_1,\sigma_2) + \varepsilon]\omega_f^{(1)}\left(\frac{1}{\sqrt{n}},\frac{1}{\sqrt{n}}\right)$$

holds;

b) this inequality is unimprovable for the class $\overline{C}(R^2)$ in the sense that for any $\varepsilon > 0$, there exist $n_1(\varepsilon) \in N$ and $fn(t,s) \in \overline{C}(R^2)$ such that for all $n > n_1(\varepsilon)$, the inequality

$$\max_{(t,s)\in G} |fn(t,s) - P_n(fn;t,s)| >$$
$$> [\,æ_1(\sigma_1,\sigma_2) - \varepsilon]\omega_{fn}\left(\frac{1}{\sqrt{n}},\frac{1}{\sqrt{n}}\right)$$

holds.

Theorem 6. If conditions (A_1), (B_1), (C_1) are satisfied, then the following relation takes place:

$$\overline{\lim_{n\to\infty}} \sup_{f\in\overline{C}(R^2)} \left\{\frac{\max_{(t,s)\in G}|f(t,s)-P_n(f;t,s)|}{\omega_f^{(1)}\left(\frac{1}{\sqrt{n}},\frac{1}{\sqrt{n}}\right)}\right\} = æ_1(\sigma_1,\sigma_2).$$

Theorem 7. If conditions (A_1), (B_1), (C_1) are satisfied, then

a) for any $\mathfrak{z}(t,s) \in \overline{C_\Omega}(R^2)$ and $\varepsilon > 0$, there exists $n_0(\varepsilon) \in N$ such that for all $n > n_0(\varepsilon)$, the relation

$$\max_{(t,s)\in G} |f(t,s) - P_n(f;t,s)| \le$$
$$\le [\,æ_2(\sigma_1,\sigma_2) + \varepsilon]\omega_f^{(2)}\left(\frac{1}{\sqrt{n}}\right) \quad \text{holds;}$$

b) this inequality is unimprovable for the class $\overline{C_\Omega}(R^2)$ in the sense that for any $\varepsilon > 0$, there exist $n_1(\varepsilon) \in N$ and a r.p. $\mathfrak{z}n(t,s) \in \overline{C_\Omega}(R^1)$ such that for all $n > n_1(\varepsilon)$, the inequality

$$\max_{(t,s)\in G}\{\,M|fn(t,s) - P_n(fn;t,s)|^2\}^{\frac{1}{2}} >$$
$$> [\,æ_2(\sigma_1,\sigma_2) - \varepsilon]\,\omega_{fn}^{(2)}\left(\frac{1}{\sqrt{n}}\right)$$

holds.

Theorem 8. Let conditions A_2), (B_2), (C_2) be satisfied. Then the following relation takes place:

$$\overline{\lim_{n\to\infty}} \sup_{f\in\overline{C}(R^2)} \left\{\frac{\max_{(t,s)\in G}|f(t,s)-P_n(f;t,s)|}{\omega_f^{(2)}\left(\frac{1}{\sqrt{n}}\right)}\right\} = æ_2(\sigma_1,\sigma_2).$$

Note that Theorems 5–8 are of independent interest and are new in the classical theory of approximation of nonrandom functions.

Let now $\mathfrak{z}(t,s) \in C_\Omega(R^2)$ be a separable sub-Gaussian continuous with probability one r.f. satisfying the condition

(A_2): $\left\|\mathfrak{z}(t,s) - \mathfrak{z}(u,v)\right\|_{sub} \le$

$$\le \omega\left(\sqrt{(t-u)^2 + (s-v)^2}\right),$$

$$(t,s),(u,v) \in R^2,$$

where $\omega(x)$ is the continuity module, for which there is an inverse function $\omega^{-1}(z)$, and the integral $\int_0^1 \frac{\omega(z)dz}{z\sqrt{|lnz|}} < \infty$, the norm $||*||_{sub}$ is introduced in [4].

With respect to the sequence of distribution functions $F_{t,s}^{(n)}(x,y)$, suppose that the following conditions are satisfied:

(B_2): partial derivatives with respect to t and s of functions $dF_{t,s}^{(n)}(x,y)$ exist and are continuous functions of (t,s)∈ Q for any $n \in N$, $(x,y) \in R^2$

$$(C_2)\text{: } [dF_{t,s}^{(n)}(x,y)]_t' = \rho_{t,s}^{n,1}(x,y)dF_{t,s}^{(n)}(x,y),$$
$$[dF_{t,s}^{(n)}(x,y)]_s' = \rho_{t,s}^{n,2}(x,y)\,dF_{t,s}^{(n)}(x,y),$$

where continuous for (t,s)∈Q functions $\rho_{t,s}^{n,i}(x,y)$, $(x,y) \in R^2$ are such that

$$\sup_{(t,s)\in Q} \int_{R^2} |\rho_{t,s}^{n,i}(x,y)|dF_{t,s}^{(n)}(x,y) < \infty, \quad i = 1,2.$$

If conditions (B_2) and (C_2) are satisfied, then for any

(t,s) ∈ Q, the partial derivatives with respect to t and s of the r.f. $P_n(\mathfrak{z};t,s)$ exist with the probability one [4, p. 268], and

$$[Pn(\mathfrak{z};t,s)]_t' = \int_{R^2} \mathfrak{z}(x,y)|\rho_{t,s}^{n,1}(x,y)|dF_{t,s}^{(n)}(x,y) \equiv P_{n,1}(\mathfrak{z};t,s),$$

$$[Pn(\mathfrak{z};t,s)]_s' = \int_{R^2} \mathfrak{z}(x,y)|\rho_{t,s}^{n,2}(x,y)|dF_{t,s}^{(n)}(x,y) \equiv P_{n,2}(\mathfrak{z};t,s).$$

Obviously, r.f.'s $P_{n,i}(\mathfrak{z};t,s)$ are sub-Gaussian, i.e.

$$M\,P_{n,i}(\mathfrak{z};t,s) = 0, \quad \sup_{(t,s)\in Q} ||Pn,i(\mathfrak{z};t,s)||_{sub} < \infty, \quad i = 1,2.$$

Assume that they satisfy the condition

(D_2): there exists a sequence of positive numbers $(\alpha_{n,i})_{n=1}^{\infty}$ such that для любого $n \in N$,

$$\sup_{(t,s)\in Q} ||Pn,i(\mathfrak{z};t,s)||_{sub} \le \alpha_{n,i}, \quad i=1,2.$$

It is not difficult to verify that the conditions (B_2), (C_2), (D_2) are satisfied for classical operators such as Bernstein, Weierstrass, Mirakyan operators, etc.

Investigate the r.f. $\eta_n(t,s) = \frac{\mathfrak{z}(t,s) - Pn(\mathfrak{z};t,s)}{c_0\,\omega\left(\frac{1}{\sqrt{n}}\right)}$ on the set G,

Where $c_0 = \sigma_1 + \sigma_2 + 1$, $\sigma_1 = \sup_{(t,s)\in G} \sigma_1(t)$, $\sigma_2 = \sup_{(t,s)\in G} \sigma_2(s)$.

Denote by d_G the diameter of the set G.

Set $c_1 = \sqrt{2ln(d_G+2)}$, $q_n = \frac{c_0\omega(d_G)}{\omega(d_G)+2d_G\sqrt{\alpha_{n,1}^2+\alpha_{n,2}^2}}$,

$$\gamma_n = \frac{1}{2}\{c_1 + \sqrt{ln\frac{1}{\omega^{-1}\left(\frac{q_n}{2}\omega\left(\frac{1}{\sqrt{n}}\right)\right)}} +$$

$$+\frac{1}{q_n\omega(\frac{1}{\sqrt{n}})}\int_0^{\omega^{-1}(\frac{q_n}{2}\omega(\frac{1}{\sqrt{n}}))}\frac{\omega(z)dz}{z\sqrt{|lnz|}}\},$$

where $\omega^{-1}(z)$ is the inverse to $\omega(x)$ function.

Theorem 9. Let conditions (A_2) – (D_2) be satisfied. Then for any $n \in N$ and for all $z \geq 64$, the inequality

$$P\{\max_{(t,s)\in G}|\frac{\eta_n(t,s)}{\gamma_n}| \geq 2z\} \leq 2exp\{-\frac{z^2}{2}\gamma_n^2\}$$

is valid.

Corollary. Let $\omega(x)$ in condition (A_2) be such that $\omega\left(\frac{1}{\sqrt{n}}\right)\gamma_n \downarrow 0$. If the conditions of Theorem 9 be satisfied, then for any $\varepsilon > 0$, $0 < \delta < 1$, for all $n \in N$,

$$n \geq n_0(\varepsilon,\delta) =$$

$$= min\{n \in N{:}2c_0\omega(\frac{1}{\sqrt{n}})(\,64\gamma_n + \sqrt{2ln\frac{2}{\delta}}) \leq \varepsilon\},$$

the inequality

$P\{\max_{(t,s)\in G}|\xi(t,s) - P_n(\xi;t,s)| < \varepsilon\} \geq 1-\delta$ holds.

It should be noted that approximations of random processes in a uniform metric is considered in the works [2], [3], [8-10] and [13].

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