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Issue Article

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FINDING AN EXPLICIT SOLUTION OF THE FREDHOLM EQUATION OF 2ND FORM IN MAPLE

***Abstract**: The theory of integral equations acts as a mathematical tool for solving various problems of mathematical modeling. The modern mathematical packages application, the dynamic development of which at the present stage is carried out successfully, creates an opportunity to simplify considerably the algorithms drawing up and to develop mathematical programs of finding the solution of integral equations at modeling of physical processes and phenomena. The article describes the development of a code for finding an explicit solution of the Fredholm integral equation of the 2nd form with a degenerate core. The method of reduction to a system of linear algebraic equations using the criterion of degeneracy of the kernel is realized.*

***Key words**: integral equation, degenerate kernel, Kramer formulas, roots of the equation.*
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Introduction

Many problems of mathematical physics are reduced to linear integral equations of the form:

$$\lambda\int_a^b K(x,t)\varphi(t)dt = f(x), a \le x \le b \quad (1)$$

$$\varphi(x) - \lambda\int_a^b K(x,t)\varphi(t)dt = f(x), a \le x \le b \quad (2)$$

concerning the unknown function $\varphi(x)$,which are called Fredholm integral equations of the 1st and 2nd kind, respectively.[1]

Here we consider a linear Fredholm equation of the 2nd form:

$$\varphi(x) - \lambda\int_a^b K(x,t)\varphi(t)dt = f(x),\ a \le x \le b, \quad (3)$$

where $K(x,t)$ is degenerate kernel.

One of the methods of finding the solution of equation (3) in explicit form is the method of reducing the original integral equation to a system of linear algebraic equations. According to the definition of the degenerate kernel, $K(x,t)$ can be represented as a finite sum of the product of functions, each of which depends on only one variable:

$$K(x,t)=\sum_{i=1}^{N} p_i(x)q_i(t), \qquad (4)$$

where each of the functions $\alpha_i(t)$, $\beta_i(t)$ are linearly independent. [2]Substituting (4) into (3), the equation takes the form:

$$\varphi(x)-\lambda\sum_{i=1}^{N} p_i(x)\int_a^b q_i(t)\varphi(t)dt = f(x).$$

Taking $C_i=\int_a^b \beta_i(t)\varphi(t)dt$, the equation is written:

$$\varphi(x)-\lambda\sum_{i=1}^{N} C_i\alpha_i(x)= f(x). \qquad (5)$$

Whence the solution of equation (3) will be represented by this formula:

$$\varphi(x)= f(x)+\lambda\sum_{i=1}^{N} C_i\alpha_i(x). \qquad (6)$$

To find the coefficients C_i, multiply equation (3) by the function $q_k(x)$ and then integrate the obtained expression within the range from a to b. As a result of these steps, equation (3) has the following form:

$$\int_a^b q_k(x)\varphi(x)dx-\lambda\sum_{i=1}^{N} C_i\int_a^b \alpha_i(x)q_k(x)\varphi(x)dx=\int_a^b f(x)q_k(x)\,dx. \qquad (7)$$

Using the notations for the integrals in (7):

$$C_k=\int_a^b f(x)q_k(x)\,dx,\ \alpha_{ik}=\int_a^b p_i(x)q_k(x)\,dx,\ f_{ik}=\int_a^b f(x)q_k(x)\,dx, \qquad (8)$$

the last expression will have the form [3], [4]:

$$C_k-\lambda\sum_{i=1}^{N} C_i\alpha_{ik}=f_k,\ k=\overline{1,N}. \qquad (9)$$

Thus, we obtain the set of equations (9), which is defined as a system of n linear equations with respect to n unknowns C_i with matrix [5]:

$$A_\lambda=\begin{pmatrix} 1-\lambda\alpha_{11} & -\lambda\alpha_{21} & \dots & -\lambda\alpha_{N1} \\ -\lambda\alpha_{12} & 1-\lambda\alpha_{22} & \dots & -\lambda\alpha_{N2} \\ \dots & \dots & \dots & \dots \\ -\lambda\alpha_{1N} & -\lambda\alpha_{2N} & \dots & 1-\lambda\alpha_{NN} \end{pmatrix}$$

If λ in equation(3) is not the same as any of the roots of the equation:

$$D(\lambda)=\det A_\lambda=\begin{vmatrix} 1-\lambda\alpha_{11} & -\lambda\alpha_{21} & \dots & -\lambda\alpha_{N1} \\ -\lambda\alpha_{12} & 1-\lambda\alpha_{22} & \dots & -\lambda\alpha_{N2} \\ \dots & \dots & \dots & \dots \\ -\lambda\alpha_{1N} & -\lambda\alpha_{2N} & \dots & 1-\lambda\alpha_{NN} \end{vmatrix}=0, \qquad (10)$$

namely $\lambda\neq\lambda_s, s=\overline{1,n}$ -system has a solution and

$$C_i=\frac{D_i(\lambda)}{D(\lambda)},\ i=\overline{1,N}, \qquad (11)$$

where the determinants $D_i(\lambda)$ are obtained from the determinant $D(\lambda)$ by replacing the i - column by the column of free terms $(f_1 \quad f_2 \quad \dots \quad f_N)^T$:

$$D_i(\lambda)=\begin{vmatrix} 1-\lambda\alpha_{11} & -\lambda\alpha_{21} & \dots f_1\dots & -\lambda\alpha_{N1} \\ -\lambda\alpha_{12} & 1-\lambda\alpha_{22} & \dots f_2\dots & -\lambda\alpha_{N2} \\ \dots & \dots & \dots & \dots \\ -\lambda\alpha_{1N} & -\lambda\alpha_{2N} & \dots f_N\dots & 1-\lambda\alpha_{NN} \end{vmatrix}.$$

Substituting the values of C_i by (11), the solution of equation (3) can be written in the form:

$$\varphi(x)=f(x)+\lambda\sum_{i=1}^{N}\frac{D_i(\lambda)}{D(\lambda)}\alpha_i(x). \qquad (12)$$

If λ in equation (3) coincides with one of the roots of equation (12): : $\lambda=\lambda_s$, which means, $D(\lambda)=0$ - the system (9) is insolvable at arbitrary right-hand sides. Therefore, integral equation (3) is insoluble at arbitrary function $f(x)$.[3]

If $D_i(\lambda_s)=0$, $D(\lambda_s)=0$, then system (9) has infinitely many solutions. The integral equation (3) will also have infinitely many solutions.[3]

An alternative method of finding a solution to equation (3) is implemented in the computer mathematics system Maple.[7] Let us implement the solution algorithm and find a solution to the Fredholm equation of the 2nd form:

$$\varphi(x)-\int_a^b(3x+2t)\varphi(t)dt=8x^2-5x,\ 0\le x\le 1$$

Calculations are performed in the *Student* package. [8] Enter the original equation:

$restart;\ with(Student[Calculus1]);$

$eq1 := \text{phi}(x) = 8\cdot x^2 - 5\cdot x + int((3\cdot x + 2\cdot t)\cdot\text{phi}(t), t = 0..1);$

$$eq1 := \phi(x) = 8x^2 - 5x + \int_0^1 (3x+2t)\,\phi(t)\,dt$$

Let us rewrite the equation in the form:

$\text{phi}(x) = 8\cdot x^2 - 5\cdot x + 3\cdot x\cdot int(\text{phi}(t), t = 0..1) + 2\cdot int(t\cdot\text{phi}(t), t = 0..1);$

$$\phi(x) = 8x^2 - 5x + 3x\left(\int_0^1 \phi(t)\,dt\right) + 2\left(\int_0^1 t\,\phi(t)\,dt\right)$$

We insert the notations $C1=\int_0^1\varphi(t)dt$,

$C2=\int_0^1 t\varphi(t)dt$, according to which the solution of Equation $\varphi(x)$ will be written in the formula:

$sol := subs(int(\text{phi}(t), t = 0..1) = C1, int(t\cdot\text{phi}(t), t = 0..1) = C2, \%);$

$$sol := \phi(x) = 3\,C1\,x + 8x^2 + 2\,C2 - 5x$$

We integrate the last equation [7]:

$e1 := int(lhs(sol), x = 0..1) = rhs(Rule['+'](Int(rhs(sol), x = 0..1)));$

$$e1 := \int_0^1 \phi(x)\,dx = \int_0^1 3x\,C1\,dx + \int_0^1 8x^2\,dx + \int_0^1 2\,C2\,dx + \int_0^1 (-5x)\,dx$$

We multiply the integrated expression $e1$ by x and integrate again in the range from to [7]:

$$e2 := int(x \cdot lhs(sol), x = 0..1) = rhs(Rule['+'](Int(expand(x \cdot rhs(sol)), x = 0..1)));$$

$$e2 := \int_0^1 x\,\phi(x)\,dx = \int_0^1 3\,C1\,x^2\,dx + \int_0^1 8\,x^3\,dx + \int_0^1 2\,C2\,x\,dx + \int_0^1 (-5\,x^2)\,dx$$

We make a system of algebraic equations to find the constants $C1$, $C2$:

$$e11 := subs(int(\mathrm{phi}(x), x = 0..1) = C1, lhs(e1)) = value(rhs(e1));$$
$$e22 := subs(int(x \cdot \mathrm{phi}(x), x = 0..1) = C2, lhs(e2)) = value(rhs(e2));$$

$$e11 := C1 = \frac{3}{2}\,C1 + \frac{1}{6} + 2\,C2$$

$$e22 := C2 = C1 + \frac{1}{3} + C2$$

Here we find the solution of the system:

$$res := solve(\{e11, e22\}, \{C1, C2\});\ assign(res);$$

$$res := \left\{C1 = -\frac{1}{3}, C2 = 0\right\}$$

Let's write down the solution of the initial equation:

sol;

$$sol;\quad \phi(x) = 8\,x^2 - 6\,x$$

Let's check the found solution:

$$simplify(lhs(eq1) - rhs(eq1));$$

$$0$$

Let us consider the reduction to a system of linear algebraic equations the Fredholm integral equation of the 2nd kind , determining directly the coefficients α_{ik} and f_k in formula (9) for C_k by formulae (8) on the example of Eq.:

$$\varphi(x) - \int_a^b (3x + 2t)\varphi(t)dt = 8x^2 - 5x .$$

The kernel $K(x,t) = 3x + 2t$ by definition of degeneracy of the kernel is represented in the form (4)[]:

$$K(x,t) = \sum_{i=1}^{2} p_i(x)q_i(t) = p_1(x)q_1(t) + p_2(x)q_2(t), \qquad (13)$$

where $p_1(x) = x$, $q_1(t) = 3$, $p_2(x) = 2$, $q_2(t) = t$.

Then by formulas (8) we calculate a_{11}, a_{12}, a_{21}, a_{22}, f_1, f_2:

$$a_{11} = \int_0^1 p_1(t)q_1(t)dt;\ a_{12} = \int_0^1 p_2(t)q_1(t)dt;\ a_{21} = \int_0^1 p_1(t)q_2(t)dt;\ a_{22} = \int_0^1 p_2(t)q_2(t)dt$$

$$f_1 = \int_0^1 q_1(t)f(t)dt;\ f_2 = \int_0^1 q_2(t)f(t)dt.$$

Let us perform the calculations in Maple[8]-[9] environment:

$$p1x := x;\ q1t := 3;\ p2x := 2;\ q2t := t;\ fx := 8\cdot x^2 - 5\cdot x;$$
$$fr := fx + lambda\cdot(C1\cdot(p1x) + C2\cdot(p2x));$$
$$a11 := int((subs(x=t, p1x))\cdot q1t, t=0..1);$$
$$a12 := int((subs(x=t, p2x))\cdot q1t, t=0..1);$$
$$a21 := int((subs(x=t, p1x))\cdot q2t, t=0..1);$$
$$a22 := int((subs(x=t, p2x))\cdot q2t, t=0..1);$$
$$f1 := int((subs(x=t, fx))\cdot q1t, t=0..1);$$
$$f2 := int((subs(x=t, fx))\cdot q2t, t=0..1);$$

$$p1x := x$$
$$q1t := 3$$
$$p2x := 2$$
$$q2t := t$$
$$fx := 8x^2 - 5x$$
$$fr := 8x^2 + x\,C1 - 5x + 2\,C2$$
$$a11 := \frac{3}{2}$$
$$a12 := 6$$
$$a21 := \frac{1}{3}$$
$$a22 := 1$$
$$f1 := \frac{1}{2}$$
$$f2 := \frac{1}{3}$$

Let us make a system to calculate the constants $C1$, $C2$ (in the system denoted by $C11, C22$, respectively) and find its solution:

$$s := (\{C11 - (a11\cdot C11 + a12\cdot C22) = f1, C22 - (a21\cdot C11 + a22\cdot C22) = f2\});$$
$$rs := solve(s, \{C11, C22\});\ assign(rs);$$

$$s := \left\{-\frac{1}{3}\,C11 = \frac{1}{3}, -\frac{1}{2}\,C11 - 6\,C22 = \frac{1}{2}\right\}$$
$$rs := \{C11 = -1, C22 = 0\}$$

Substituting the values of $C11$ and $C22$ with the subs command, the solution of the equation is written:

$$req := subs(C1 = C11, C2 = C22, fr);$$
$$req := 8x^2 - 6x\ .$$

For the complete solution of the integral equation, both in the first and in the second case, the analytical study of the degeneracy of the kernel $K(x,t)$ was not carried out: the degeneracy of the kernel was accepted as a fact. In order that the mathematical realization of the validity of finding the solution of the equation is carried out, we use the criterion of degeneracy of the kernel of the integral equation.[10]. The wording of the criterion is as follows "For the kernel $K(x,t)$ to be degenerate it is necessary and enough that $\det A_n[K](x,t) \neq 0$, $\det A_m[K](x,t) = 0$, for $\forall m > n$, where the matrix $A_m[K](x,t)$ is of the form:

$$A_m[K](x,t) = \begin{pmatrix} K & K'_x & \dots & K^{(m-1)}_{x^{(m-1)}} \\ K'_y & K''_{xy} & \dots & K^{(m)}_{x^{(m1)}y} \\ \dots & \dots & \dots & \dots \\ K^{(m-1)}_{y^{(m-1)}} & K^{(m)}_{xy^{(m1)}} & \dots & K^{(2m-2)}_{x^{(m1)}y^{(m-1)}} \end{pmatrix} »[10] \qquad (13)$$

Let us realize the criterion for the kernel $K(x,t) = 3x + 2t$. We construct the matrices $A_2[K](x,t)$ and $A_3[K](x,t)$, by computing the coefficients, according to (13), then calculate the values of the determinants $A_2[K](x,t)$ и $A_3[K](x,t)$[8]-[9]:

```
k11 := Kxt; k12 := diff(Kxt, x$1); k13 := diff(Kxt, x$2);
k21 := diff(Kxt, t$1); k22 := diff(Kxt, x$1, t$1); k23 := diff(Kxt, x$2, t$1);
k31 := diff(Kxt, t$2); k32 := diff(Kxt, x$1, t$2); k33 := diff(Kxt, x$2, t$2);
DK2 := Matrix(2, 2, [k11, k12, k21, k22]); zDK2 := Determinant(DK2);
DK3 := Matrix(3, 3, [k11, k12, k13, k21, k22, k23, k31, k32, k33]); zDK3
    := Determinant(DK3);
```

$$DK2 := \begin{bmatrix} 3x + 2t & 3 \\ 2 & 0 \end{bmatrix}$$

$$zDK2 := -6$$

$$DK3 := \begin{bmatrix} 3x + 2t & 3 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$zDK3 := 0$$

As we see, the value of the determinant $A_2[K](x,t)$ is different from zero: $zDK2 = -6$, the value of the determinant $A_3[K](x,t)$ is zero $zDK3 = 0$.

Let's compose a conditional ***if*** statement:

```
if zDK2 ≠ 0 and zDK3 = 0 then print ('Yadro_uravneniya_virogdenno_');
else print ('Yadro_uravneniya_NE_virogdenno');
 fi;
```

*print Yadro_uravneniya_virogdenno_*

Knowing that the kernel of the equation is degenerate, we find the solution of the integral equation[8]-[9] in the loop:

```
if zDK2 ≠ 0 and zDK3 = 0 then
 p1x := x; q1t := 3; p2x := 2; q2t := t ; fx := 8·x^2 − 5·x;
fr := fx + lambda·(C1·(p1x) + C2·(p2x));
a11 := int((subs(x = t, p1x))·q1t, t = 0 ..1);
a12 := int((subs(x = t, p2x))·q1t, t = 0 ..1);
a21 := int((subs(x = t, p1x))·q2t, t = 0 ..1);
a22 := int((subs(x = t, p2x))·q2t, t = 0 ..1);
f1 := int((subs(x = t, fx))·q1t, t = 0 ..1);
f2 := int((subs(x = t, fx))·q2t, t = 0 ..1);
s := ({C11 − (a11·C11 + a12·C22) = f1, C22 − (a21·C11 + a22·C22) = f2});
 rs := solve(s, {C11, C22}); assign(rs);
req := subs(C1 = C11 , C2 = C22, fr);
 fi;
```

The code of the solution of the integral equation (3) is:

```
restart; with(LinearAlgebra) :
fx := 8·x^2 − 5·x; Kxt := 3·x + 2·t; lambda := 1;
eq := phi(x) = fx + lambda·int(Kxt·phi(t), t = 0 ..1);
k11 := Kxt; k12 := diff(Kxt, x$1); k13 := diff(Kxt, x$2);
k21 := diff(Kxt, t$1); k22 := diff(Kxt, x$1, t$1); k23 := diff(Kxt, x$2, t$1);
k31 := diff(Kxt, t$2); k32 := diff(Kxt, x$1, t$2); k33 := diff(Kxt, x$2, t$2);
DK2 := Matrix(2, 2, [k11, k12, k21, k22]); zDK2 := Determinant(DK2);
DK3 := Matrix(3, 3, [k11, k12, k13, k21, k22, k23, k31, k32, k33]); zDK3
     := Determinant(DK3);
if zDK2 ≠ 0 and zDK3 = 0 then print ('Yadro_uravneniya_virogdenno_');
else print ('Yadro_uravneniya_NE_virogdenno');
fi;
if zDK2 ≠ 0 and zDK3 = 0 then
 p1x := x; q1t := 3; p2x := 2; q2t := t ; fx := 8·x^2 − 5·x;
fr := fx + lambda·(C1·(p1x) + C2·(p2x));
a11 := int((subs(x = t, p1x))·q1t, t = 0 ..1);
a12 := int((subs(x = t, p2x))·q1t, t = 0 ..1);
a21 := int((subs(x = t, p1x))·q2t, t = 0 ..1);
a22 := int((subs(x = t, p2x))·q2t, t = 0 ..1);
f1 := int((subs(x = t, fx))·q1t, t = 0 ..1);
f2 := int((subs(x = t, fx))·q2t, t = 0 ..1);
s := ({C11 − (a11·C11 + a12·C22) = f1, C22 − (a21·C11 + a22·C22) = f2});
 rs := solve(s, {C11, C22}); assign(rs);
req := subs(C1 = C11 , C2 = C22, fr);
 fi;
```

As can be seen, the function $f(x)$, the kernel $K(x,t)$, λ are inserted at the initial stage. Then the integral equation (3) is written down. In the second cycle we insert $p_1(x)=x$, $q_1(t)=3$, $p_2(x)=2$, $q_2(t)=t$, according to the representation $K(x,t)$ in the form (13). The solution of the integral equation is performed in the *LinearAlgebra* package. Approbation of the code on numerous examples of the form (3) allowed us to automate the solution of the Fredholm integral equation with a degenerate kernel[11].

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