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Y.R. Krakhmaleva
M.Kh.Dulaty Taraz Regional University
Candidate of Technical Sciences,
yuna_kr@mail.ru

B. Mirzayan
M.Kh.Dulaty Taraz Regional University
Master's student,
bibarysmyrzayan@gmail.com

FINITE-DIFFERENCE EULER METHOD FOR SOLVING PROBLEMS OF VARIATIONAL CALCULATION IN MAPLE ENVIRONMENT

Abstract*: The articelle describes the realization of the finite-difference Euler method on the example of a simple problem of variational calculus in the system of analytical calculations. Difficulties and ways of solution arising at realization of the method are shown.*

Key words*: functional, broken line, approximate solution, exact solution.*

Language*: English*

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Introduction

Methods of direct search for the functional minimum in the calculus of variations are an important tool for solving a certain class of problems. They are aimed at finding extrema of functionals defined on functions and can be applied in various fields. The basic idea of direct methods is as follows. Let it be necessary to find the minimum of some functional $V[y]$ defined on some class M of admissible curves. For the problem to make sense, we must assume that there exist curves in the class M for which the functional $V[y]$ is finite. As a consequence, the exact lower bound of $\inf\limits_{y\notin M} V[y] = \mu > -\infty$. is also finite. Then, according to the definition of the exact lower bound, we can say that there exists a sequence of curves $\{y_n\} \in M$, such that $\lim\limits_{n\to\infty} V[y_n] = \mu$. In this case, the limit curve $\{y_n\}$ is defined for $y^{(0)}$ and the limit transition is satisfied:

$$V[y^{(0)}] = \lim_{n\to\infty} V[y_n],$$

then we have

$$V[y^{(0)}] = \mu .$$

This means that the limit curve $y^{(0)}$ and will be the solution of the problem [1].

Thus, to solve the variational problem by the direct method, perform the following steps:

1. construct the minimizing sequence $\{y_n\}$;

2. prove the existence of a limit curve at the sequence $\{y_n\}$ using theoretical knowledge from functional analysis and topology. Depending on the properties of M and the norm in which convergence is considered, different approaches are considered (compactness, closedness theorems, weak convergence, etc.).

3. prove the legitimacy of the limit transition. The most difficult and final stage, which requires the proof of the continuity of the functional.

The members of the minimizing sequence can be regarded as approximate solutions of the corresponding variational problem.[1]-[2]

The main difficulty in the rigorous justification of direct methods of calculus of variations is that minimizing sequences may not converge to any limit function even when the existence of a solution is certain. The possibility that the functional may not be continuous with respect to the chosen norm or topology is not excluded. It should be noted an important fact, the dependence of the application of mathematical apparatus in choosing the class and norm in which the convergence is searched and analyzed. These complexities make direct methods a powerful, but often difficult to implement tool.[1],[3]

In solving the problem of finding the extremum of a functional, it is well known that the following methods are used

$$V[y]=\int_a^b F(x,y,y')dx\,, \qquad (1)$$

consider the problem of finding the extremum of a function of a finite number of variables. For this purpose, the desired function is replaced by a polyline with vertices - fixed abscissas of points $x_1, x_2, ..., x_n$, the derivative of $y'(x)$ function - by the difference relation:

$$y'(x)\approx\frac{y(x_{m+1})-y(x_m)}{\Delta x_m}. \qquad (2)$$

Let us consider the finite-difference Euler method on the example of a simple variational problem: we need to find the extremum of a functional:

$$V[y]=\int_a^b F(x,y,y')dx\,, y(a)=y_a\,, y(b)=y_b \qquad (3)$$

Approximate solutions to problem (3) are broken lines, which are composed of a given number of n links with vertices:

$$x_i=a+i\Delta x\,,$$

where $\Delta x=\frac{b-a}{n}$. On such broken lines we consider a function $\Phi(y_1,y_2,...,y_{n-1})$, whose variables are the unknown ordinates $y_1,y_2,...,y_{n-1}$ of the vertices of the broken line. By constructing a system of equations:

$$\begin{cases}\dfrac{\partial\Phi(y_1,y_2,...,y_{n-1})}{\partial y_1}=0,\\ \dfrac{\partial\Phi(y_1,y_2,...,y_{n-1})}{\partial y_2}=0,\\ \quad ...\\ \dfrac{\partial\Phi(y_1,y_2,...,y_{n-1})}{\partial y_{n-1}}=0.\end{cases}$$

The ordinates of the vertices of the polyline are determined. At $y_1,y_2,...,y_{n-1}$, the function $\Phi(y_1,y_2,...,y_{n-1})$ reaches extrema.[1],[2]-[5]

Example . Find an approximate solution to the problem of the minimum of the functional[1]:

$$V[y]=\int_0^1\left((y')^2+2y\right)dx, y(0)=0, y(1)=0.$$

Let's solve the example in Maple system by the code [6]:

$restart; with(linalg) : interface(displayprecision=3) :$

When calculating approximations, the operations are usually performed on decimal numbers. In such calculations, Maple preserves 10 significant floating point digits, which can be reduced with the *interface(displayprecisions=n)* command. In our example . $N=3$.

We create a procedure for replacing the integrand function by finite differences, using formula (2) for $y'(x)$, to $y(x)\approx y(x_m)$:

$$F:=\mathbf{proc}(Y,m,h)\ \frac{(Y[m+1]-Y[m])^2}{h^2}+2\cdot Y[m]\,\mathbf{end\ proc}:$$

$$F:=\mathbf{proc}(Y,m,h)\ \ (Y[m+1]-Y[m])\text{^}2/h\text{^}2+2*Y[m]\ \ \mathbf{end\ proc}$$

A procedure is created to replace this integral by a sum using the rectangle formula[6]-[7]:

$$\int_a^b f(x)dx\approx\left(f(a)+f(x_1)+...+f(x_{n-1})\right)\cdot\Delta x$$

$$JN:=\mathbf{proc}(h,F,N)\,\mathbf{options}\ operator, arrow;\ \ h\cdot(sum(F(Y,i,h), i=0\,..\,N-1))\ \ \mathbf{end\ proc};$$

$$JN := (h, F, N) \rightarrow h\left(\sum_{i=0}^{N-1} F(Y, i, h)\right)$$

The limits of integration are set. The number of nodal points is fixed and the formula for the integration step is entered:

$$a := 0 : b := 1 : N := 5 : h := \frac{(b-a)}{N} :$$

A loop is constructed to calculate the abscissas of the vertices (points) of the polyline:

for j **from** 0 **to** N **do** $X[j] := h \cdot j + a$: **end do**;

$$X_0 := 0$$
$$X_1 := \frac{1}{5}$$
$$X_2 := \frac{2}{5}$$
$$X_3 := \frac{3}{5}$$
$$X_4 := \frac{4}{5}$$
$$X_5 := 1$$

We write the formula for the functional $V[y]$ as a function of the ordinates of the vertices of the polyline[6]:

Phi $:= JN(h, F, N)$;

$$\Phi := 5\left(Y_1 - Y_0\right)^2 + \frac{2}{5} Y_0 + 5\left(Y_2 - Y_1\right)^2 + \frac{2}{5} Y_1 + 5\left(Y_3 - Y_2\right)^2 + \frac{2}{5} Y_2 + 5\left(Y_4 - Y_3\right)^2 + \frac{2}{5} Y_3 + 5\left(Y_5 - Y_4\right)^2 + \frac{2}{5} Y_4$$

The boundary conditions are fixed:

$Y[0] := 0$; $Y[\mathrm{N}] := 0$;

$$Y_0 := 0$$
$$Y_5 := 0$$

The equations of the minimizing system for determining the ordinates are compiled Y_1, Y_2, Y_3, Y_4 [6]:

for k **to** $N-1$ **do** $eq[k] := evalf(diff(\mathrm{Phi}, Y[k])) = 0$; **end do**;

$$eq_1 := 20.000\, Y_1 - 10.000\, Y_2 + 0.400 = 0$$
$$eq_2 := 20.000\, Y_2 - 10.000\, Y_1 - 10.000\, Y_3 + 0.400 = 0$$
$$eq_3 := 20.000\, Y_3 - 10.000\, Y_2 - 10.000\, Y_4 + 0.400 = 0$$
$$eq_4 := 20.000\, Y_4 - 10.000\, Y_3 + 0.400 = 0$$

Then, as a result of the performed actions, the minimizing system of eqns equations with unknowns is constructed Y_1, Y_2, Y_3, Y_4 [6]:

```
var := { } : eqns := { } :
for k to N − 1 do
var := var union {Y[k]};
eqns := eqns union {eq[k]};
end do;
```

$$var := \{Y_1\}$$

$$eqns := \{20.000\,Y_1 - 10.000\,Y_2 + 0.400 = 0\}$$

$$var := \{Y_1, Y_2\}$$

$$eqns := \{20.000\,Y_1 - 10.000\,Y_2 + 0.400 = 0,\ 20.000\,Y_2 - 10.000\,Y_1 - 10.000\,Y_3 + 0.400 = 0\}$$

$$var := \{Y_1, Y_2, Y_3\}$$

$$eqns := \{20.000\,Y_1 - 10.000\,Y_2 + 0.400 = 0,\ 20.000\,Y_2 - 10.000\,Y_1 - 10.000\,Y_3 + 0.400 = 0,\ 20.000\,Y_3 - 10.000\,Y_2 - 10.000\,Y_4 + 0.400 = 0\}$$

$$var := \{Y_1, Y_2, Y_3, Y_4\}$$

The left side of each equation of the system contains not only the unknowns Y_1, Y_2, Y_3, Y_4, with coefficients, but also numerical values without a definite unknown, that is, essentially representing the free terms of the equations. Applying the *solve* command to the system *eqns*:

$res := solve(eqns, var);\ assign(res):$

We have:

Let's write the command like this:

$$res1 := \{Y_1 = -0.080,\ Y_2 = -0.120,\ Y_3 = -0.120,\ Y_4 = -0.080\}$$

The same values are obtained if you follow these steps. Let's write out a system of equations:

The system consists of 4-equations with 4-unknowns. Let's make the matrix A of the system and the column of $B1$ free terms. [8]To do this, let's isolate the first equation of the system and then write the left side of the equation as a polynomial :

Let's define the coefficients at unknowns in $p1$ as elements of the system matrix :

Then we partition the polynomial $p1$ and determine the position of $b1$ according to the partitioning :

According to the partition $b1$ is the third element of $p1$:

We write identical steps for the 2nd, 3rd, and 4th equations:

Having defined the elements of the system matrix and the column of free terms, let us compose A and $B1$ [9]:

Now we find Y_1, Y_2, Y_3, Y_4:

Generate a list of vertex points of the polyline and output the polyline $L5$ [6] :

Approximate solution:

We find the exact solution to the problem[6];

The graph of constructing the approximate $L5$ and exact solution y_x of the problem has the following form [8]-[10]:

As we can see, the exact solution differs markedly from the approximate solution obtained by the Euler method.

For comparison, let us consider the implementation of the described code for the functional[6]:

$$V[y]=\int_{-1}^{1}\left((y')^2-2y'e^x+\cos x\right)dx,\; y(-1)=2,\; y(1)=3.$$

Minimizing system of equations for determining ordinates Y_1, Y_2, Y_3, Y_4 has the form:

We find a solution to the system:

An approximate solution to the problem is a polyline $L5$:

Prescribe the actions described above, as a result of which the matrix A of the system and the matrix $B1$ of the free terms are defined as follows:

Then the ordinates of the vertices of the polyline Y_1, Y_2, Y_3, Y_4:

The approximate solution of the problem is written in the form of a polyline $L55$:

Approximate solution:

The exact solution of the problem is of the form:

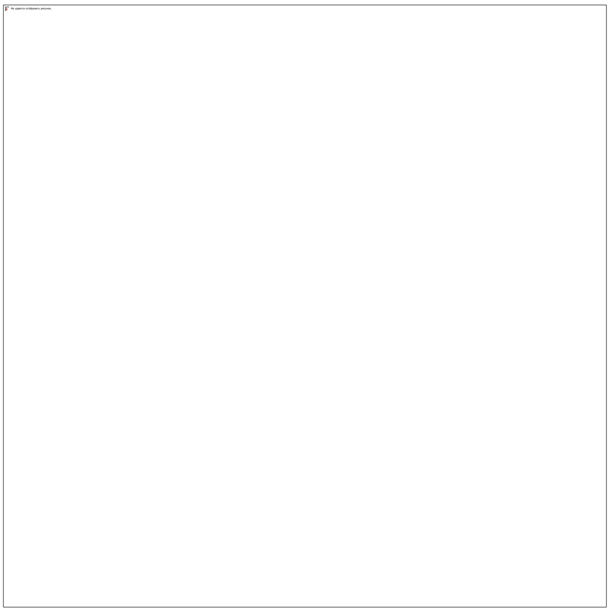
For comparison, let us plot the graph of the approximate $L5$, $L55$ and the exact solution of the y_x problem [8]-[10] :

It can be seen that the result has a noticeable improvement over the result obtained for the Example above, which confirms the fact that the accuracy of the approximate solution depends on the expression of the integrand.

Increasing the integration step up to $N = 7$, we have an approximate solution in the form of a broken line $L7$ and $L77$:

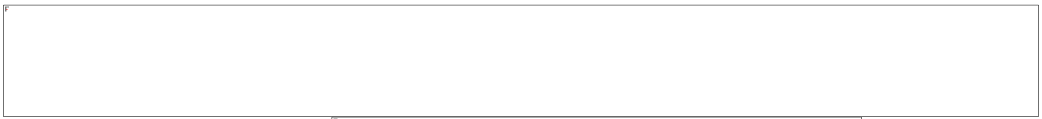
Approximate solution:

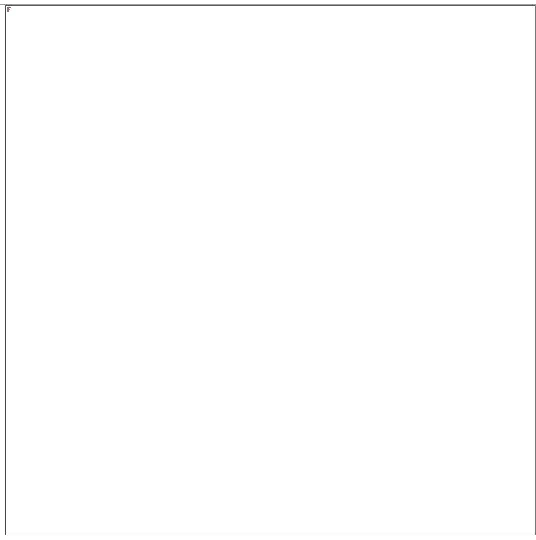
The graph in this case is as follows:



As you can see, the approximate solutions in both cases have a slight difference. But the approximate solution obtained by the solve team is more accurate

Let's compare the results of the solution at steps $N=5, N=7$ with the exact solution by plotting the graphs [10]:





When the steps are increased, the accuracy of the computation of the approximate solution increases, as one would expect. At the same time, the number of equations in the system of equations with unknowns $Y_1, Y_2, ..., Y_n$ increases to $N-1$. At the same time, the number of operations to determine $Y_1, Y_2, ..., Y_n$ increases. The main task of the finite-difference Euler method is to find the unknowns $Y_1, Y_2, ..., Y_n$. According to the prescribed code for finding the unknowns, the Maple program handles the task without much difficulty.

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