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APPLICATION OF THE LAPLACE TRANSFORM TO SOLVE THE VOLTAIRE EQUATION OF THE 2ND FORM

Abstract: *One of the alternative methods of solving integral equations is considered to be finding their solution in computer mathematics systems. The advantages possessed by SCMs open new possibilities for conducting research in this environment. The article describes the development of a code for constructing the resolvent of the generalized Voltaire integral equation of the 2nd form with its subsequent solution in Maple.*

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Introduction

Integral equations represent an essential mathematical tool to solve various problems of mathematical modeling and complementing in this field the apparatus of differential equations. Consequently, finding the solution of integral equations is among the topical issues.

Equations of the form

$$\varphi(x)-\lambda\int_a^x K(x-t)\varphi(t)dt = f(x), a \le x. \quad (1)$$

as well as analogous equations of the form

$$\int_a^x K(x-t)\varphi(t)dt = f(x), \ a \le x, \quad (2)$$

represent an important special class of Voltaire integral equations, which are usually called convolution type equations because the operation

$$\{K,\varphi\} = \int_a^x K(x-t)\varphi(t)dt$$

is the convolution of 2 functions K and φ. [1]

We can take the lower limit of integration to be zero in (1), (2): $a = 0$. In fact, by substituting the variable $x - a = \xi$, $t - a = \eta$ we arrive at the equation:

$$\varphi(a+\xi)-\lambda\int_0^{\xi}K(\xi-\eta)\varphi(\eta)d\eta=f(\xi),\xi\geq 0.$$

Let us consider an equation of the form

$$\varphi(x)-\lambda\int_a^{x}K(x-t)\varphi(t)dt=f(x),x\geq 0,\quad (3)$$

which is called the Voltaire equation of the 2nd kind with a kernel depending on the difference of arguments.[1],[2]

The main means of studying such equations is the Laplace transform. Let belong $f(t)$ to the class of functions , for which the integral $f(t)$

$$\int_0^{\infty}|f(t)|e^{-\eta t}dt$$

converges if η is chosen sufficiently large positive. Then the direct Laplace transform takes place:

$$F(p)=\int_0^{\infty}f(t)e^{-pt}dt,$$

inverse Laplace transform:

$$f(t)=\frac{1}{2\pi i}\int_{s-i\infty}^{s+i\infty}F(p)e^{pt}dp.$$

Inasmuch as this transformation transforms the convolution into an ordinary product under some constraints. Thus, the solution of the integral equation reduces to the inversion of the Laplace transform.[1],[3]

Applying the Laplace transform to equation (3), the result is:

$$\overline{\varphi}-\lambda\overline{K}\overline{\varphi}=\overline{f},$$
$$\overline{\varphi}(1-\lambda\overline{K})=\overline{f},$$
$$\overline{\varphi}=\frac{\overline{f}}{1-\lambda\overline{K}}.\quad (4)$$

In (4), we insert the notations of the Laplace transform of the functions[1],[4]:

$$\overline{\varphi}=\int_0^{\infty}\varphi(x)e^{-px}dx,\ \overline{K}=\int_0^{\infty}K(t)e^{-pt}dt,$$
$$\overline{f}=\int_0^{\infty}f(x)e^{-px}dx.\quad (5)$$

Reversing equation (5) we find:

$$\varphi(x)=\frac{1}{2\pi i}\int_L\frac{\overline{f}(p)}{1-\lambda\overline{K}(p)}e^{px}dp,$$

where the line 123 is located to the right of the special points of the integrand. Let us consider the transformation of the obtained solution. Let us write formula (4) in the form[1]:

$$\overline{\varphi}=\frac{\overline{f}}{1-\lambda\overline{K}}=\overline{f}\frac{(1-\lambda\overline{K})+\lambda\overline{K}}{1-\lambda\overline{K}}=\overline{f}+\frac{\lambda\overline{K}}{1-\lambda\overline{K}}\overline{f}.$$

Let's denote,

$$\overline{R}_{\lambda}=\frac{\overline{K}}{1-\lambda\overline{K}}.\quad (6)$$

Then

$$\overline{\varphi}=\overline{f}+\lambda\overline{R_{\lambda}}\,\overline{f}.$$

Considering $\overline{R}_{\lambda}$ as the Laplace transform of some function[4]-[5]:

$$\overline{R_{\lambda}}=\int_0^{\infty}R_{\lambda}(t)e^{-pt}dt,$$

may be written

$$\varphi(x)=f(x)+\lambda\int_a^{x}R_{\lambda}(x-t)f(t)dt.\quad (7)$$

Thus, if knowing the function $\overline{R}_{\lambda}$, then formula (6) gives the solution of problem[1],[6]- [8].

According to the above theory, let us find the solution of the equation :

$$\varphi(x)-\lambda\int_0^{x}\sin(x-y)\varphi(y)dy=e^{-x}.\quad (8)$$

We will solve the equation in Maple. Connect the package of integral transformations *inttrans* and enter the initial equation [9]:

```
restart; with(inttrans);
eq1 := phi(x) = exp(-x) + lambda·int(sin(x - y)·phi(y), y = 0 ..x);
```

$$eq1:=\phi(x)=e^{-x}+\lambda\left(\int_0^{x}\sin(x-y)\,\phi(y)\,dy\right)$$

Let us write down the kernel of the $K(x,y)$ equation. Then we introduce the notation for the kernel, according to the substitution of the difference of arguments $x-y$:

$$Kxy := \sin(x-y);\ Kt := subs(x-y=t, Kxy);$$

$$Kxy := \sin(x-y)$$
$$Kt := \sin(t)$$

We use the direct Laplace transform on the variable t.[5] This is possible using the function $laplace(\exp, t, p)$, where $\exp$ - is the expression to be transformed, 123 is the variable with respect to which the original equation is written, and p – is the variable on which the result of the transformation will be written. We find the Laplace transform of the kernel and get the image $Kt = \sin t$:

$$KtL := laplace(Kt, t, p);$$

$$KtL := \frac{1}{p^2+1}$$

Knowing the image of the kernel, we find the image for the resolvent by the formula (6):

$$R := \frac{KtL}{1 - \text{lambda}\cdot KtL};$$

$$R := \frac{1}{\left(p^2+1\right)\left(-\frac{\lambda}{p^2+1}+1\right)}$$

Let's transform the image of the resolvent R:

$$RR := simplify\left(\frac{KtL}{1-\text{lambda}\cdot KtL}\right);$$

$$RR := \frac{1}{p^2-\lambda+1}$$

To find the original resolvent we will use the function $invlaplace(\exp, p, t)$, where $\exp$ - is an equation with respect to the variable p, t - is the variable with respect to which the resulting dependence is written[10]:

$$OR := invlaplace(RR, p, t)$$

$$OR := \frac{\sin\left(\sqrt{-1+\lambda}\ t\right)}{\sqrt{-1+\lambda}}$$

Find the original of the resolvent with respect to the original variables:

$$ORxy := subs(t = x-y, OR);$$

$$ORxy := \frac{\sin\left(\sqrt{-1+\lambda}\ (x-y)\right)}{\sqrt{-1+\lambda}}$$

By substituting the obtained resolvent, the solution of the equation has the form: $eq2 := subs(Kxy = ORxy, eq1);$

$$eq2 := \phi(x) = e^{-x} + \lambda\left(\int_0^x \frac{\sin\left(\sqrt{-1+\lambda}\ (x-y)\right)\phi(y)}{\sqrt{-1+\lambda}}\,dy\right)$$

The solution is valid for any λ and is an integer function of λ. To find the solution to the equation at $\lambda = 1$ we need to solve for the uncertainty using decomposition.

Let us see how the resolvent changes for specific values of λ. To do this, just enter the value λ at the beginning of the solution and we will get not only a representation of the resolvent, but also a formula for solving the original equation at the entered value λ. Assuming $\lambda = 3$, we have:

$$lambda := 3;$$
$$eq1 := \text{phi}(x) = \exp(-x) + lambda\cdot int(\sin(x-y)\cdot\text{phi}(y), y = 0..x);$$
$$Kxy := \sin(x-y);$$
$$Kt := subs(x-y=t, Kxy);$$
$$KtL := laplace(Kt, t, p);$$
$$R := \frac{KtL}{1-\text{lambda}\cdot KtL};$$
$$RR := simplify\left(\frac{KtL}{1-\text{lambda}\cdot KtL}\right);$$
$$OR := invlaplace(RR, p, t);$$

$$ORxy := subs(t = x-y, OR);\ eq2 := subs(Kxy = ORxy, eq1);$$

$$\lambda := 3$$

$$eq1 := \phi(x) = e^{-x} + 3\left(\int_0^x \sin(x-y)\,\phi(y)\,dy\right)$$

$$Kxy := \sin(x-y)$$

$$Kt := \sin(t)$$

$$KtL := \frac{1}{p^2+1}$$

$$R := \frac{1}{\left(p^2+1\right)\left(-\frac{3}{p^2+1}+1\right)}$$

$$RR := \frac{1}{p^2-2}$$

$$OR := \frac{1}{2}\sin\left(\sqrt{2}\,t\right)\sqrt{2}$$

$$ORxy := \frac{1}{2}\sin\left(\sqrt{2}\,(x-y)\right)\sqrt{2}$$

$$eq2 := \phi(x) = e^{-x} + 3\left(\int_0^x \frac{1}{2}\sin\left(\sqrt{2}\,(x-y)\right)\sqrt{2}\,\phi(y)\,dy\right)$$

As we can see, the resolvent of the equation has the form:

$$R = \frac{\sqrt{2}\sin\sqrt{2}(x-y)}{2}.$$

Let us find a solution to the last equation:

$$\varphi(x) - 3\int_0^x \frac{\sqrt{2}\sin\sqrt{2}(x-y)}{2}\varphi(y)dy = e^{-x}. \quad (7)$$

Enter the equation:

$$eq2 := \text{phi}(x) = \exp(-x) + 3\cdot int\left(\frac{\text{sqrt}(2)}{2}\sin(\text{sqrt}(2)\cdot(x-y))\cdot\text{phi}(y), y = 0..x\right);$$

$$eq2 := \phi(x) = e^{-x} + 3\left(\int_0^x \frac{1}{2}\sqrt{2}\sin\left(\sqrt{2}\,(x-y)\right)\phi(y)\,dy\right)$$

Let us apply the direct Laplace transform for equation $eq2$:

$$laplace(eq2, x, p);$$

$$laplace(\phi(x), x, p) = \frac{1}{1+p} + \frac{3}{2}\,\frac{laplace(\phi(x), x, p)}{\frac{1}{2}p^2+1}.$$

Find the equation to represent the equation:

$$subs(laplace(\text{phi}(x), x, p) = \text{Phi}, \%);$$

$$\Phi = \frac{1}{1+p} + \frac{3\Phi}{p^2+1}$$

We find a solution for the transformant:

$$solve(\%, \text{Phi});$$

$$\frac{p^2+1}{(1+p)\left(p^2-2\right)}$$

Using the inverse Laplace transform to represent the transformant, we find the original of the desired function[9],[10]:

$invlaplace(\%, p, x);$

$$\frac{3}{4}e^{x} - \frac{1}{4}e^{-x}(6x-1)$$

Thus, the required function of equation (7) has the form:

$\text{phi} := unapply(\%, x);$

$$\phi := x \to \frac{3}{4}e^{x} - \frac{1}{4}e^{-x}(6x-1).$$

Let us perform a verification of the solution:

$leq2 := lhs(eq2);\ req2 := rhs(eq2);\ simplify(\ leq2 - req2);$

$$leq2 := \frac{3}{4}e^{x} - \frac{1}{4}e^{-x}(6x-1)$$

$$req2 := e^{-x} + \frac{3}{4}\left(e^{2x} - 2x - 1\right)e^{-x}$$

$$0$$

The solution has been found correctly. Collecting the commands in one group, we obtain the code for constructing the resolvent of the integral equation (3) and the solution of the initial equation itself. Taking into account that the equation is entered at the initial stage, we have an automated code that has been tested on various examples. Speed and minimal time spent on finding the solution are the advantages of the code and its use in applied problems.

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