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Yu.R. Krakhmaleva
M.Kh. Dulaty Taraz Regional University
d.t.s,
yuna_kr@mail.ru

A.U. Nurimbetov
M.Kh. Dulaty Taraz Regional University
d.t.s,
alibek55@mail.ru

B. Mirzayan
M.Kh. Dulaty Taraz Regional University
Master student,
bibarysmyrzayan@gmail.com

RITZ METHOD FOR SOLVING PROBLEMS OF CALCULUS OF VARIATIONS IN MAPLE

Abstract: *The paper describes the Ritz method realization on the example of the simplest problem of calculus of variations in the system of analytical calculations. The graphical realization of the method is given.*
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Introduction

The method, the principle of which is to construct a minimizing sequence, is called the Ritz method. The method was proposed in 1909 and is named after its founder, the German mathematician Walter Ritz. One of the important fields of application of the Ritz method is optics. For example, it is used to calculate the characteristics of optical resonances.

As established, the solution of equation :

$$\mathrm{B}u = f(P), \qquad (1)$$

where B - is a positive operator, is represented in the definition of the minimum value of a functional of the form:

$$f[u] = (\mathrm{B}u, u) - 2(u, f). \qquad (2).$$

In finding an approximate solution to an equation, the following steps are performed. Select functions whose sequence belongs to the set of functions continuous on $[a,b]$ and having continuous first and second derivatives in it [1]:

$$\varphi_1(P), \varphi_2(P), ..., \varphi_n(P), ..., \qquad (3)$$

It is necessary that for the functions $\{\varphi_n(P)\}$ the conditions are satisfied:

1) there is completeness in energy for $\{\varphi_n(P)\}$;

2) functions from $\{\varphi_n(P)\}$ were linearly independent at $\forall n$.

The functions (3) are called coordinate (basis) functions.

For the first terms $\{\varphi_n(P)\}$ is a linear combination of [1]-[3]:

$$u_n(P)=\sum_{j=1}^{n} a_j\varphi_j(P), \qquad (4)$$

where a_j - random numbers.

Substituting (4) into (2) yields an expression of the functional $F[u(P)]$ as a function with n variables $a_1, a_2, \ldots, a_n$:

$$F[u_n]=\left(\sum_{j=1}^{n} a_j \mathrm{B}\varphi_j, \sum_{j=1}^{n} a_j\varphi_j, f\right)-2\left(\sum_{j=1}^{n} a_j\varphi_j, f\right)=\sum_{j=1}^{n}\sum_{k=1}^{n}(\mathrm{B}\varphi_j,\varphi_k)a_j a_k-2\sum_{j=1}^{n}(\varphi_j, f)a_j . \quad (5)$$

Depending on the values of a_j, the function (5) takes different values. Select a_j so that $F[u_n]$ in (5), has the smallest value. The necessary conditions are met for the minimum of $F[u_n]$:

$$\frac{\partial F[u_n]}{\partial a_i}=0, \qquad (6)$$

$i=1,2,\ldots,n$.

When the operator B is positive, the minimum value of $F[u_n]$ is obtained under the condition that a_j is a solution of the system (6).

Calculating the derivatives of $\frac{\partial F[u_n]}{\partial a_i}$, $i=1,2,\ldots,n$, and substituting (2.8) into 123 , we obtain the explicit system (9):

$$\frac{\partial F[u_n]}{\partial a_i}=\sum_{j=1}^{n}\sum_{k=1}^{n}(\mathrm{B}\varphi_j,\varphi_k)\left[\frac{da_j}{da_i}a_k+a_j\frac{da_k}{da_i}\right]-2\sum_{j=1}^{n}(\varphi_j, f)\frac{da_j}{da_i}=$$

$$=\sum_{j=1}^{n}\sum_{k=1}^{n}(\mathrm{B}\varphi_j,\varphi_k)[\delta_{ji}a_k+a_j\delta_{ki}]-2\sum_{j=1}^{n}(\varphi_j, f)\delta_{ji},$$

where $\delta_{ji}=\begin{cases}1, & j=i\\ 0, & j\neq i\end{cases}$ - Kronecker symbol.

Hence, the expression for $\frac{\partial F[u_n]}{\partial a_i}$ is written as:

$$\frac{\partial F[u_n]}{\partial a_i}=\sum_{k=1}^{n}(\mathrm{B}\varphi_i,\varphi_k)a_k+\sum_{j=1}^{n}(\mathrm{B}\varphi_j,\varphi_i)a_j-2(\varphi_i, f)=2\sum_{k=1}^{n}(\mathrm{B}\varphi_i,\varphi_k)a_k-2(\varphi_i, f), \qquad (9)$$

in which operator B is symmetric: $(\mathrm{B}\varphi_j,\varphi_i)=(\mathrm{B}\varphi_i,\varphi_j)$.

Substituting expression (10) into (9) forms a system of equations called the Ritz system [1],[4]:

$$\sum_{k=1}^{n}(\mathrm{B}\varphi_i,\varphi_k)a_k=(\varphi_i, f), \quad i=1,2,\ldots,n. \quad (11)$$

The determinant of the system matrix (11) is the Gramm determinant of linearly independent functions a_j. Its value is not equal to zero. So, it always has a solution (11) at positive B. When the system is solved, the a_k is calculated. The found values of a_k, are substituted into (4) and $u_n(P)$ is determined. The approximate solution of equation (1), obtained by Ritz method is $u_n(P)$ [1],[5].

Theorem: The minimizing sequence for (2) is only the approximate solutions of equation (1) by the Ritz method if the solution of equation (4) is finite in energy.

It turns out that convergence to the exact solution of approximate solutions by the Ritz method is in energy. There is also convergence of solutions in the mean under the condition of positive definiteness of the operator B.

Let us consider finding an approximate solution of the minimum functional problem by the Ritz method[1]:

$$V[y]=\int_{-1}^{1}\left((y')^2-2y'e^x+\cos x\right)dx,\ y(-1)=2,\ y(1)=3.$$

Let us perform the solution in the Maple computer mathematics system. It is necessary to select basis functions, being guided by the fact that for basis functions $\varphi_n(x)$ the determination of their completeness by energy with respect to the class of functions from the set of functions continuous on $[a,b]$ and having continuous first and second derivatives in it, taking as B the corresponding operator[6].

When comparing $u_N(x)$ at certain N values, the correctness of the result obtained is evaluated. For this purpose, $u_N(x)$ at a particular value N is taken as the final result. Such a technique can also be used in function selection $\varphi_n(x)$ [1].

Let us uppose the function $u(x)$ has a representation in the form of a series:

$$u(x)=\sum_{n=0}^{\infty}a_n\varphi_n(x).$$

At least, by pre-specifying the accuracy, it is possible to obtain an approximation of $u(x)$ by a segment of a series. Obviously, the functions $\{\varphi_n(x)\}$ approximate quite accurately any inferred function and its derivative. This implies that condition 1) for $\{\varphi_n(x)\}$ Ritz's method is satisfied. This condition represents a sufficient condition by considering a minimizing sequence. In addition, condition 2) for $\{\varphi_n(x)\}$ of the Ritz method requires the functions to be linearly independent. The fulfillment of the condition of unambiguous solvability of the system of equations (3) allows this to be achieved. A mandatory condition for functions $\{\varphi_n(x)\}$ is the fulfillment of the boundary conditions of the problem[1],[6].

As functions $\{\varphi_n(x)\}$ in many cases we consider functions of the form:

$$\varphi_k(x)=x^{k-1}(x-a)^m(x-b)^m, \qquad (12)$$

where $k=1,2,3,\ldots$.

The completeness condition on the energy of the species operator is satisfied for the set (12):

$$(-1)^m\frac{d^{2m}u}{dx^{2m}} \qquad (13)$$

with boundary conditions:

$$u^{(k)}(a)=u^{(k)}(b)=0,1,\ldots,m-1. \qquad (14)$$

In individual cases, there may be a system of functions of the form:

$$\varphi_k(x)=\sin\left[\frac{k\pi(x-a)}{(b-a)}\right], \qquad (15)$$

where $k=1,2,3,\ldots$

Let the basis functions be the system of functions (15). Let us introduce an approximating function Us [1]:

```
restart; with(plots) :
phi0 := x→y1 + ((y2 − y1)·(x − x1))/(x2 − x1);
phi := (x, n)→sin((n·Pi·(x − x1))/(x2 − x1));
Us := proc(x, N) option operator, arrow; local n;
phi0(x) + sum('a[n]·phi(x, n)', 'n'= 1 ..N);
end proc;
```

$$\phi0 := x\rightarrow y1+\frac{(y2-y1)\,(x-x1)}{x2-x1}$$

$$\phi := (x,n)\rightarrow \sin\left(\frac{n\,\pi\,(x-x1)}{x2-x1}\right)$$

$$Us := (x, N) \to \phi 0(x) + \sum_{'n'=1}^{N} 'a_n \, \phi(x, n)'$$

The assignment of function Us is carried out in the procedure. By assigning specific values to N we have an explicit expression of the functions, in which the terms of the series according to N appear. For example,

$Us(x, 1);\ Us(x, 2);$

$$y1 + \frac{(y2 - y1)(x - x1)}{x2 - x1} + a_1 \sin\left(\frac{\pi (x - x1)}{x2 - x1}\right)$$

$$y1 + \frac{(y2 - y1)(x - x1)}{x2 - x1} + a_1 \sin\left(\frac{\pi (x - x1)}{x2 - x1}\right) + a_2 \sin\left(\frac{2\pi (x - x1)}{x2 - x1}\right)$$

The procedure for formulating the Ritz equations is as follows [1]:

```
Ritz :=proc(F, u, i0, N, a)local Fu, eqns, var, eq, i, res;
global x1, x2;
Fu := simplify(int(subs(y(x) = u, F), x = x1 ..x2));
eqns := { } :  var := { } :
for i from i0  to N do
var := var union {a[i]};
eq[i] := diff(Fu, a[i]) = 0;
eqns := eqns union {eq[i]} :
od;
res := solve(eqns, var);
assign(res);
end proc;
interface(displayprecision = 3) :
```

The boundary points, the number of terms of the series, and the integrand function are entered:

```
x1 := -1; x2 := 1; y1 := 2; y2 := 3;
N := 5 : c1 :='cross': c2 :='circle':
c3 :='box': c4 :='point': c :='diamond':
y :='y':
F := (diff(y(x), x))^2 - 2·(diff(y(x), x))·exp(x) + cos(x);
for j from 1  to N do
a := array(1 ..j) :
 Ritz(F, Us(x, j), 1, j, a);
 end do:
```

$$x1 := -1$$
$$x2 := 1$$
$$y1 := 2$$
$$y2 := 3$$
$$F := \left(\frac{d}{dx} y(x)\right)^2 - 2\left(\frac{d}{dx} y(x)\right) e^x + \cos(x)$$

The approximating function $Us(x,1)$ has the following form, according to the performed calculations:

$Ritz(F, Us(x, 1), 1, 2, a);$

$$Us1(x,1) := \frac{5}{2} + \frac{1}{2}x - 0.566\sin\left(\frac{1}{2}\pi(x+1)\right)$$

The exact solution to the problem:

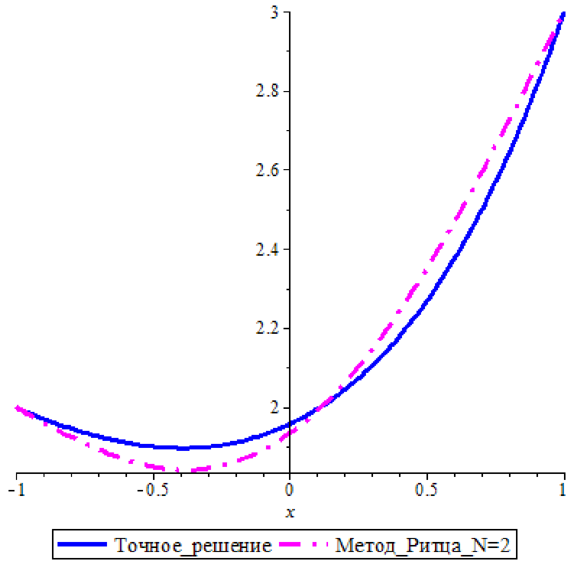
$y :=$ **proc**(x)**option** *operator*, *arrow*;

$$e^{x} + \left(\frac{1}{2} + \frac{1}{2}e^{-1} - \frac{1}{2}e\right)x + \frac{5}{2} - \frac{1}{2}e - \frac{1}{2}e^{-1}$$

end proc;

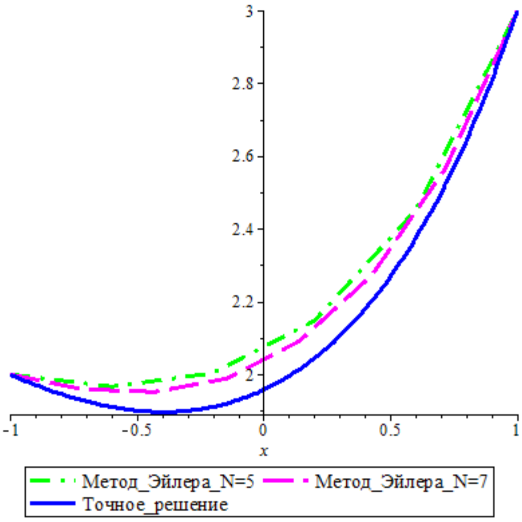
Let's plot the graphs of the approximate solution $Us(x,1)$ and the exact solution $y(x)$ [9],[10]:

plot([$y(x)$, *pu_1*], $x = x1..x2$, linestyle = [*solid*, *dashdot*], *color* = [blue, magenta], thickness = 3, numpoints = 150, *legend* = ["Точное_решение", "Метод_Ритца_N=2"]);



Results of solving the original problem using Euler's method at steps $N = 5, N = 7$ with exact solution:

plot([L5, L7, y_x], x = −1 .. 1, color = [green, magenta, blue], linestyle = [*dashdot*, *dash*, *solid*], thickness = 3, legend = ["Метод_Эйлера_N=5", "Метод_Эйлера_N=7", "Точное_решение"]);



As we see, the approximate solution constructed by the Ritz method with smaller steps gives a better solution than the solution constructed by the Euler method with larger steps.

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