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**SECTION 1. Theoretical research in mathematics.**

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# SOME PROPERTIES OF THE LATTICE OF F-CLOSED RIGHT IDEALS

***Abstract***: *Throughout this paper R is a unitary associative ring and f is an injective ring endomorphiesm of R. In the present article, we introduce the notion of the lattice Lat(R, f ) of all f-closed right ideals of R with some special operation instead of the intersection operation. The paper is devoted to the study of this lattice. In particular, we investigate the interrelationship between the lattice of all f-closed right ideals of R and the lattice of right ideals of the Cohn-Jordan extension A. We obtained some results in this direction.*

*In Theorem 1 we give necessary and sufficient conditions, in terms of the lattice Lat(R, f ), for the Cohn-Jordan extension A be a right Artinian ring. This theorem implies in particular that A is right Artinian provided that R is right Artinian. Theorem 2 is a structural theorem and states that a ring R with a bounded length of chains of the right f-closed ideals is embeddable in a semisimple Artinian ring. The authors' original proof is based on the Cohn-Jordan extension. The Cohn-Jordan extensions were first introduced in [8] for the study of skew polynomial rings constructed by means of a ring endomorphism. Five open questions are formulated.*

***Key words***: *lattice, composition length, right Artinian rings*

***Language***: *English*

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**Introduction**

Throughout this paper all rings are associative. In what follows let $R$ be a ring and $f$ be an injective ring endomorphism of $R$. Recall that David Alan Jordan introduced in [8] the construction of the smallest ring $A$ containing $R$ such that every endomorphism $f$ of $R$ to an automorphism of $A$ (see also [2, 10]). More precisely, let $A = A(R,f)$ be a ring, containing $R$ and $\tilde{f}$ be an automorthism of $A$ that extends the endomorphism $f$. Then the ring $A$ together with the automorphism $\tilde{f}$ is called the Cohn-Jordan extension of the ring $R$ with endomorphism $f$, if each element $a$ of $A$ can be presented as $a = f^{-n}(r)$, where $r \in R$ and $n$ is some positive integer.

**Materials and Methods**

Using the construction of a direct limit one can verify that this extension and is unique. Let us consider a countable number of copies $R_i$ of the ring $R$ labeled by nonnegative integers $i$ and natural isomorphisms $\varepsilon_i : R \to R_i$. Given a pair of indexes $(m, n)$ with $m \le n$, the mapping $f_{m,n} : R_m \to R_n$ is defined by $f_{m,n} = \varepsilon_n \circ f^{n-m} \circ \varepsilon_m^{-1}$. Then the equality $f_{m,n} = f_{m,k} \circ f_{k,n}$ holds for all $k$ such that $m \le k \le n$. Therefore, there is a direct limit

$$A(R,f) = \varinjlim (R_n, f_{m,n} : m,n \ge 0),$$

One can check that the mapping defined by $\tilde{f} : \varepsilon_i(r) \mapsto \varepsilon_{i+1}(r)$, where $i \ge 0, r \in R$, is a correctly defined automorphism of $A(R,f)$ and the restriction of $\tilde{f}$ to $R$ is equal to the endomorphism $f$. Thus, the direct limit $A(R,f)$ and it's automorphism $\tilde{f}$ form the Cohn-Jordan extension of $R$ with respect to the endomorphism $f$.

There is another method to construct the Cohn-Jordan extension of $R$. This method based on the classical left ring of quotients $Q = X^{-1}R[x,f]$, where $X = \{1, x, x^2, x^3, \dots\}$ and the multiplication in the skew polynomial ring is defined by $xr = f(r)$ $(\forall r \in R)$. It is easy to prove that the set $A = \bigcup_{n\geq 0} x^{-n}Rx^n$ of all elements $Q$ of the form $x^{-n}rx^n$ is a ring containing $R$. Furthermore, the inner automorphism $\tilde{f} : x^{-n}rx^n \mapsto x^{1-n}rx^{n-1}$ of $A$ is an extension of the endomorphism $f$. Moreover, $A = \bigcup_{n\geq 0} \tilde{f}^{-n}(R)$. Cohn-Jordan extensions are studied and used for differ purposes in scientific papers [9, 11].

Throughout the sequel, let $A$ together with $\tilde{f}$ denote the Cohn-Jordan extension of the ring $R$ and its injective endomorphism $f$.

*Definition* 1. A right ideal $I$ of $R$ is said to be $f$-closed (see [3, 7]), if

$$I = \bigcup_{n=1}^{\infty} f^{-n}\left(f^n(I)R\right).$$

One can check, that a right ideal $I$ of $R$ is $f$-closed if and only if $I = IA \cap R$. It implies that any $f$-closed right ideal $I$ of $R$ has the form $I = MA \cap R$ for some available right ideal M of $A$. Conversely, all the right ideals of this kind are f-closed.

An ideal $N$ of $R$ is called an $f$-ideal if $f^{-1}(N) = N$ (see [1, 5]).

Let us consider the lattice $\mathrm{Lat}(R,f)$ of all $f$-closed right ideals of $R$ supplied the following operations:

1) $B \wedge C = B \cap C$;

2) $B \vee C = \bigcup_{n\geq 0} f^{-n}\left(f^n(B)R + f^n(C)R\right)$.

The result of the first operation is the largest $f$-closed right ideal contained in the $f$-closed right ideals $B$ and $C$. The result of the second operation is the smallest $f$-closed right ideal containing both right ideals $B$ and $C$.

*Remark* a). If $B$ and $C$ are $f$-closed right ideals of $R$, then the following two equalities hold:

$$f^{-n}(f^n(B)R \cap f^n(C)R) \subseteq f^{-n}(f^n(B)R) = B$$

and $f^{-n}(f^n(B)R \cap f^n(C)R) \subseteq f^{-n}(f^n(C)R) = C$.

Therefore, we need not to describe the operation $B \wedge C$ in the same way as the operation $B \vee C$, because $\bigcup_{n\geq 0} f^{-n}\left(f^n(B)R \cap f^n(C)R\right) = B \cap C$.

б). The following relation holds:

$$B \vee C = (BA + CA) \cap R.$$

Recall that, the submodules of some right module $M_R$ over a ring $R$, partially ordered by inclusion, form a modular lattice. In particular, the lattice of right ideals of some ring is a modular lattice. This means that the lattice satisfies the following condition called "Modular law": if $B$, $C$ and $D$ are submodules of a module $M$ over a ring $R$ and $B$ □ C, then $(C \cap D) + B = C \cap (D + B)$.

*Proposition* 1. Let $B$, $C$ и $D$ are $f$-closed right ideals of $R$ with $B \subseteq C$. Then $B \vee (C \cap D) \subseteq C \cap (B \vee D) \subseteq (BA + (CA \cap DA)) \cap R$.

*Proof.* First we show that "$B \vee (C \cap D) \subseteq C \cap (B \vee D)$".

Let $r \in B \vee (C \cap D) = (BA + (C \cap D)A) \cap R$. Then $r = \sum b_i a_i + \sum x_j \tilde{a}_j$, where $b_i \in B$, $x_j \in C \cap D$, $a_i, \tilde{a}_j \in A$. Observe, that $\sum b_i a_i \in BA \subseteq CA$ and $\sum x_j \tilde{a}_j \in CA$. Hence $r \in CA \cap R = C$. Since $b_i \in B$, $x_j \in D$, we have $r \in (BA + DA) \cap R = B \vee D$. Therefore, $r \in C \cap (B \vee D)a$.

Next we show that "$C \cap (B \vee D) \subseteq (BA + (CA \cap DA)) \cap R$". To prove this inclusion observe that $BA \subseteq CA$ and by modular law we obtain that $C \cap (B \vee D) \subseteq CA \cap (BA + DA) = BA + (CA \cap DA)$. QED.

*Corollary* 1. If $B \vee (C \cap D) = (BA + (CA \cap DA)) \cap R$ for all $f$-closed right ideals of $R$, then the lattice $\mathrm{Lat}(R,f)$ of all $f$-closed right ideals of $R$ is modular.

*Lemma* 1. If $M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots \subsetneq M_d$ is a strictly ascending chain of right ideals of $A$ of the length $d$, then $R$ must have strictly ascending chain of $f$-closed right ideals of length $d$.

*Proof.* Choose elements $m_i \in M_i \backslash M_{i-1}$ ($i$ = 1, 2, …, $d$). By Definition 1 $m_i \in f^{-n_i}(R)$ for some non-negative integers $n_1, n_2, \dots, n_d$. Let $n$ be the largest of these integer numbers. Then $b_i = f^n(m_i) \in R$ for all $i$ = 1, 2, …, $d$ and right ideals $B_i = \tilde{f}^n(M_i)$ form the strictly ascending chain $B_0 \subsetneq B_1 \subsetneq B_2 \subsetneq \cdots \subsetneq B_d$. Moreover, $b_i \in B_i \cap R$ и $b_i \notin B_{i-1} \cap R$ for all $i$ = 1, 2, …, $d$. Hence the chain of $f$-closed right ideals of $R$

$$B_0 \cap R \subsetneq B_1 \cap R \subsetneq B_2 \cap R \subsetneq \cdots \subsetneq B_d \cap R$$

is strictly ascends. But this chain has length $d$. QED.

*Lemma* 2. Let $B_0 \subsetneq B_1 \subsetneq B_2 \subsetneq \cdots \subsetneq B_d$ be a strictly ascending chain of $f$-closed right ideals of $R$ of length $d$. Then

$$B_0A \subsetneq B_1A \subsetneq B_2A \subsetneq \cdots \subsetneq B_dA$$

is a strictly ascending chain of right ideals of $A$ of the same length $d$.

*Proof.* If the relation $B_{i-1}A = B_iA$ were satisfied at some point in the second chain, then we would have

$$B_{i-1} = B_{i-1}A \cap R = B_iA \cap R = B_i$$

by virtue of the $f$-closeness of the right ideals $B_{i-1}$ and $B_i$. But the last equality contradicts the condition of the lemma. QED.

*Theorem* 1. The following conditions (1) and (2) are equivalent:

(1) $A$ is right Artinian;

(2) there exists a non-negative integer $d$ such that all strictly ascending chains of $f$-closed right ideals of $R$ have length at most $d$.

*Proof.* "(1)⇒(2)". Let $A$ be right Artinian. Then by Hopkins–Levitzki theorem $A$ is also right Noetherian and by Jordan-Holder theorem (see [6], Theorem 4.10, P. 44) $A$ has finite composition length $d$ (as a right module over itself). If $\mathrm{Lat}(R, f)$ contained a strictly ascending chain of $f$-closed right ideals of length more than d, then by Lemma 2 the ring A would contain a chain of right ideals of length more than $d$. This leads to a contradiction. Therefore, all strictly ascending chains of $f$-closed right ideals of $R$ have length at most $d$.

"(2)⇒(1)". Suppose that condition (2) holds. Then Lemma 1 shows that lengths of all strictly ascending chains of right ideals of $A$ do not exceed $d$. It follows that $A$ is right Artinian of length at most $d$. QED.

*Proposition* 2. Let be an endomorphism of $S$ and $N$ be an $F$-ideal of $S$. Suppose that $\mathrm{Ker}\, F \subseteq N$. Then $F: S \to S$ induces the endomorphism $f: S/N \to S/N$ such that $f(s+N) = F(s) + N$ for all $s \in S$. In addition, the diagram

$$\begin{array}{ccc} S & \xrightarrow{F^n} & S \\ \pi \downarrow & & \downarrow \pi \\ R & \xrightarrow{f^n} & R \end{array}$$

is commutative in the following sense:

a) $\pi \circ F^n(s) = f^n \circ \pi(s)$ for all positive integer $n$ and all $s \in S$;

b) if $Y$ is an ideal of $S$ and $N \subseteq Y$, then $\pi\big(F^{-n}(Y)\big) = f^{-n}\big(\pi(Y)\big)$.

*Proof.* a).

$f^n \circ \pi(s) = f^n(s+N) = F^n(s) + N = \pi \circ F^n(s)$.

Check equality b):

$$\pi\big(F^{-n}(Y)\big) = \{x + N \in R : F(x) \in Y\} =$$
$$= \{x + N \in R : f(x+N) \in \pi(Y)\} =$$
$$= \{x + N \in R : F(x) + N \in \pi(Y)\} =$$
$$= \{x + N \in R : f(x+N) \in \pi(Y)\} = f^{-n}\big(\pi(Y)\big).$$

Let $S$ be a ring and $N$ be a prime radical of $S$. QED.

*Lemma* 3. Let $S$ be a ring satisfying ascending chain condition on right annihilators. Suppose that every nil-subring of $R$ is nilpotent. Let $F$ be an endomorphism of $S$ with $\mathrm{Ker}\, F \subseteq N$. Than $f^{-1}(N) = N$.

For a proof we refer on [2; 4].

*Theorem* 2. Let $F$ be an endomorphism of $S$ and $\mathrm{Ker}\, fF \subseteq N$. Suppose that $d$ there exists a non-negative integer $d$ such that all strictly ascending chains of $F$-closed right ideals of $S$ have length at most $d$. Then the quotient-ring $R = S/N$ can be embedded in a product of finitely many matrix rings over division rings $D_i$.

*Proof.* The right annihilator of a set in the ring $S$ is the intersection of S and the right annihilator of this set in the Cohn-Jordan extension $A = A(S, F)$, i.e.

$$r_S(M) = S \cap r_{A(S,F)}(M).$$

It follows that all right annihilators in the ring $S$ are $F$-closed. Hence, by Theorem 1, the ring $S$ is a subring of the right Artinian ring, and every nil subring of an Aritinian ring is nilpotent. Therefore, every nil subring of $S$ is nilpotent.

Let $P$ be a prime radical of $A(S, F)$ and $\tilde{F}$ be an automorphism of $A(S, F)$ extending $F$. Then $P$ is a nilpotent ideal and, consequently, $P \cap \tilde{F}^{-n}(S) \subseteq \mathrm{rad}\left(\tilde{F}^{-n}(S)\right) = \tilde{F}^{-n}(N)$.

Thus $= \bigcup_{n=0}^{\infty} \tilde{F}^{-n}(N)$. It implies $N \subseteq P$. Moreover, since $P$ is a nilpotent ideal of $S$, it follows that $P \cap S \subseteq N$. Therefore, $P \cap S = N$. By Proposition 2 the last equality shows that the map

$$s + N \mapsto s + P \ (s \in S)$$

is an embedding of the quotient-ring $R = S/N$ in the semisimple Artinian ring $A(S, F)/P$. To complete the proof of the theorem, it remains to note that the ring $A(S, F)/P$ is isomorphic to a finite direct product of complete matrix rings over the division rings by Weddeburn-Artin theorem (see [6], § 61, Theorem 5.16). QED.

*Proposition* 2. A right ideal $L$ of $A$ is essential in $A$ if and only if for each nonzero element $r \in R$ и for every nonnegative integer $n$, there is a nonnegative integer $m$ such that $f^m(r)R \cap \tilde{f}^{n+m}(L) \neq 0$.

*Proof.* Let $L$ be an essential right ideal of the ring $A$, let $0 \neq r \in R$ and let $n$ be a nonnegative integer. Set $a = \tilde{f}^{-n}(r)$. Since $aA \cap L \neq 0$, there is a number $m \geq 0$ such that $a \cdot \tilde{f}^{-n-m}(R) \cap L \neq 0$. Applying the automorphism $\tilde{f}^{n+m}$ to the last inequality, the demanded inequality $f^m(r)R \cap \tilde{f}^{n+m}(L) \neq 0$ follows.

Suppose now that for each nonzero element $r \in R$ and for any nonnegative integer $n$ there is a nonnegative integer $m$ such that $f^m(r)R \cap \tilde{f}^{n+m}(L) \neq 0$. Every element $a \in A$ can be represented in the form $a = \tilde{f}^{-n}(r)$ where $r \in R$ and $n \geq 0$. Applying the automorphism $\tilde{f}^{-n-m}$ to the inequality $f^m(r)R \cap \tilde{f}^{n+m}(L) \neq 0$, we get that $a\tilde{f}^{-n-m}(R) \cap L \neq 0$. QED.

**Conclusion**

Here are some problems which will probably be useful for magistrates and graduate students.

*Open problems:*

1. Give necessary and sufficient conditions on $R$ and $f$ for the lattice $\mathrm{Lat}(R, f)$ be modular. Give some examples demonstrating that these conditions are essential.

2. If $\mathrm{Lat}(R, f)$ satisfies the descending chain condition, then does A need to be right Artinian?

3. Suppose that $\mathrm{Lat}(R, f)$ contains some chain of length $d$, and all strictly ascending chains of $f$-closed right ideals of $R$ have length at most $d$. Is it true that all maximal strictly ascending chains of $f$-closed right ideals have length $d$?

4. What is the relationship between the essential elements of the lattice $\mathrm{Lat}(R, f)$ and the essential right ideals of the Cohn-Jordan extension?

5. Find a necessary and sufficient condition for the Cohn-Jordan extension $A(R, f)$ to be a right Goldie ring.

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